Solutions to Problems in Jackson, *Classical Electrodynamics*, Third Edition

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December 8, 1999

Chapter 2

Problem 2.1

A point charge q is brought to a position a distance d away from an infinite plane conductor held at zero potential. Using the method of images, find:

(a) the surface-charge density induced on the plane, and plot it;

(b) the force between the plane and the charge by using Coulomb's law for the force between the charge and its image;

(c) the total force acting on the plane by integrating $\sigma^2/2\epsilon_0$ over the whole plane;

(d) the work necessary to remove the charge q from its position to infinity;

(e) the potential energy between the charge q and its image (compare the answer to part d and discuss).

(f) Find the answer to part d in electron volts for an electron originally one angstrom from the surface.

(a) We'll take d to be in the z direction, so the charge q is at $(x, y, z) = (0, 0, d)$. The image charge is $-q$ at $(0, 0, -d)$. The potential at a point $\mathbf{r}$ is

$$\Phi(\mathbf{r}) = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{|\mathbf{r} - d\mathbf{k}|} - \frac{1}{|\mathbf{r} + d\mathbf{k}|}\right]$$

The surface charge induced on the plane is found by differentiating this:

$$\begin{aligned}
\sigma &= -\epsilon_0 \frac{d\Phi}{dz}\Big|_{z=0} \\
&= -\frac{q}{4\pi}\left[\frac{-(z-d)}{|\mathbf{r}+d\mathbf{k}|^3} + \frac{(z+d)}{|\mathbf{r}+d\mathbf{k}|^3}\right]\Big|_{z=0} \\
&= -\frac{qd}{2\pi(x^2+y^2+d^2)^{3/2}} \qquad (1)
\end{aligned}$$

We can check this by integrating this over the entire xy plane and verifying that the total charge is just the value $-q$ of the image charge:

$$\begin{aligned}
\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sigma(x,y)dxdy &= -\frac{qd}{2\pi}\int_0^{\infty}\int_0^{2\pi} \frac{rd\psi dr}{(r^2+d^2)^{3/2}} \\
&= -qd\int_0^{\infty} \frac{rdr}{(r^2+d^2)^{3/2}} \\
&= -\frac{qd}{2}\int_{d^2}^{\infty} u^{-3/2}du \\
&= -\frac{qd}{2}\left|-2u^{-1/2}\right|_{d^2}^{\infty} \\
&= -q \qquad \surd
\end{aligned}$$

(b) The point of this problem is that, for points above the z axis, it doesn't matter whether there is a charge $-q$ at $(0,0,d)$ or an infinite grounded sheet at $z=0$. Physics above the z axis is exactly the same whether we have the charge or the sheet. In particular, the force on the original charge is the same whether we have the charge or the sheet. That means that, if we assume the sheet is present instead of the charge, it will feel a reaction force equal to what the image charge would feel if it were present instead of the sheet. The force on the image charge would be just $F = q^2/16\pi\epsilon_0 d^2$, so this must be what the sheet feels.

(c) Total force on sheet

$$\begin{aligned}
&= \frac{1}{2\epsilon_0}\int_0^{\infty}\int_0^{2\pi} \sigma^2 dA \\
&= \frac{q^2d^2}{4\pi\epsilon_0}\int_0^{\infty} \frac{rdr}{(r^2+d^2)^3} \\
&= \frac{q^2d^2}{8\pi\epsilon_0}\int_{d^2}^{\infty} u^{-3}du \\
&= \frac{q^2d^2}{8\pi\epsilon_0}\left|-\frac{1}{2}u^{-2}\right|_{d^2}^{\infty} \\
&= \frac{q^2d^2}{8\pi\epsilon_0}\left[\frac{1}{2}d^{-4}\right]
\end{aligned}$$

$$= \frac{q^2}{16\pi\epsilon_0 d^2}$$

in accordance with the discussion and result of part b.

(d) Work required to remove charge to infinity

$$\begin{aligned} &= \frac{q^2}{4\pi\epsilon_0}\int_d^\infty \frac{dz}{(z+d)^2} \\ &= \frac{q^2}{4\pi\epsilon_0}\int_{2d}^\infty u^{-2}du \\ &= \frac{q^2}{4\pi\epsilon_0}\frac{1}{2d} \\ &= \frac{q^2}{8\pi\epsilon_0 d} \end{aligned}$$

(e) Potential energy between charge and its image

$$= \frac{q^2}{8\pi\epsilon_0 d}$$

equal to the result in part d.

(f)

$$\begin{aligned} \frac{q^2}{8\pi\epsilon_0 d} &= \frac{(1.6\,\cdot\,10^{-19}\,\text{coulombs}\,)^2}{8\pi(8.85\,\cdot\,10^{-12}\,\text{coulombs}\,\text{V}^{-1}\text{m}^{-1})(10^{-10}\,\text{m}\,)} \\ &= 7.2\,\cdot\,(1.6\,\cdot\,10^{-19}\,\text{coulombs}\,\cdot\,1\,\text{V}\,) \\ &= 7.2\ \text{eV}\,. \end{aligned}$$

Problem 2.2

Using the method of images, discuss the problem of a point charge q *inside* a hollow, grounded, conducting sphere of inner radius a. Find

(a) the potential inside the sphere;

(b) the induced surface-charge density;

(c) the magnitude and direction of the force acting on q.

(d) Is there any change in the solution if the sphere is kept at a fixed potential V? If the sphere has a total charge Q on its inner and outer surfaces?

Problem 2.3

A straight-line charge with constant linear charge density λ is located perpendicular to the $x-y$ plane in the first quadrant at (x_0, y_0). The intersecting planes $x = 0, y \geq 0$ and $y = 0, x \geq 0$ are conducting boundary surfaces held at zero potential. Consider the potential, fields, and surface charges in the first quadrant.

(a) The well-known potential for an isolated line charge at (x_0, y_0) is $\Phi(x, y) = (\lambda/4\pi\epsilon_0)\ln(R^2/r^2)$, where $r^2 = (x-x_0)^2 + (y-y_0)^2$ and R is a constant. Determine the expression for the potential of the line charge in the presence of the intersecting planes. Verify explicitly that the potential and the tangential electric field vanish on the boundary surface.

(b) Determine the surface charge density σ on the plane $y = 0, x \geq 0$. Plot σ/λ versus x for $(x_0 = 2, y_0 = 1), (x_0 = 1, y_0 = 1)$, and $(x_0 = 1, y_0 = 2)$.

(c) Show that the total charge (per unit length in z) on the plane $y = 0, x \geq 0$ is

$$Q_x = -\frac{2}{\pi}\lambda \tan^{-1}\left(\frac{x_0}{y_0}\right)$$

What is the total charge on the plane $x = 0$?

(d) Show that far from the origin [$\rho \gg \rho_0$, where $\rho = \sqrt{x^2+y^2}$ and $\rho_0 = \sqrt{x_0^2 + y_0^2}$] the leading term in the potential is

$$\Phi \to \Phi_{\text{asym}} = \frac{4\lambda}{\pi\epsilon_0}\frac{(x_0)(y_0)(xy)}{\rho^4}.$$

Interpret.

(a) The potential can be made to vanish on the specified boundary surfaces by pretending that we have three image line charges. Two image charges have charge density $-\lambda$ and exist at the locations obtained by reflecting the original image charge across the x and y axes, respectively. The third image charge has charge density $+\lambda$ and exists at the location obtained by reflecting the original charge through the origin. The resulting potential in the first quadrant is

$$\begin{aligned} \Phi(x, y) &= \frac{\lambda}{4\pi\epsilon_0}\left(\ln\frac{R^2}{r_1^2} - \ln\frac{R^2}{r_2^2} - \ln\frac{R^2}{r_3^2} + \ln\frac{R^2}{r_4^2}\right) \\ &= \frac{\lambda}{2\pi\epsilon_0}\ln\frac{r_2 r_3}{r_1 r_4} \end{aligned} \quad (2)$$

where

$$r_1^2 = [(x-x_0)^2 + (y-y_0)^2] \qquad r_2^2 = [(x+x_0)^2 + (y-y_0)^2]$$

$$r_3^2 = [(x-x_0)^2 + (y+y_0)^2] \qquad r_4^2 = [(x+x_0)^2 + (y+y_0)^2].$$

From this you can see that

- when $x = 0$, $r_1 = r_2$ and $r_3 = r_4$
- when $y = 0$, $r_1 = r_3$ and $r_2 = r_4$

and in both cases the argument of the logarithm in (2) is unity.

(b)

$$\begin{aligned}\sigma &= -\epsilon_0 \frac{d}{dy}\Phi \\ &= -\frac{\lambda}{2\pi}\left(\frac{1}{r_2}\frac{dr_2}{dy} + \frac{1}{r_3}\frac{dr_3}{dy} - \frac{1}{r_1}\frac{dr_1}{dy} - \frac{1}{r_4}\frac{dr_4}{dy}\right)\Big|_{y=0}\end{aligned}$$

We have $dr_1/dy = (y-y_0)/r_1$ and similarly for the other derivatives, so

$$\begin{aligned}\sigma &= -\frac{\lambda}{2\pi}\left(\frac{y-y_0}{r_2^2} + \frac{y+y_0}{r_3^2} - \frac{y-y_0}{r_1^2} - \frac{y+y_0}{r_4^2}\right)\Big|_{y=0} \\ &= -\frac{y_0\lambda}{\pi}\left(\frac{1}{(x-x_0)^2+y_0^2} - \frac{1}{(x+x_0)^2+y_0^2)}\right)\end{aligned}$$

(c) Total charge per unit length in z

$$\begin{aligned}Q_x &= \int_0^\infty \sigma dx \\ &= -\frac{y_0\lambda}{\pi}\left[\int_0^\infty \frac{dx}{(x-x_0)^2+y_0^2} - \int_0^\infty \frac{dx}{(x+x_0)^2+y_0^2}\right]\end{aligned}$$

For the first integral the appropriate substitution is $(x - x_0) = y_0 \tan u$, $dx = y_0 \sec^2 u du$. A similar substitution works in the second integral.

$$\begin{aligned} &= -\frac{\lambda}{\pi}\left[\int_{\tan^{-1} - \frac{x_0}{y_0}}^{\pi/2} du - \int_{\tan^{-1}\frac{x_0}{y_0}}^{\pi/2} du\right] \\ &= -\frac{\lambda}{\pi}\left[\frac{\pi}{2} - \tan^{-1}\frac{-x_0}{y_0} - \frac{\pi}{2} + \tan^{-1}\frac{x_0}{y_0}\right] \\ &= -\frac{2\lambda}{\pi}\tan^{-1}\frac{x_0}{y_0}. \end{aligned} \tag{3}$$

The calculations are obviously symmetric with respect to x_0 and y_0. The total charge on the plane $x = 0$ is (3) with x_0 and y_0 interchanged:

$$Q_y = -\frac{2\lambda}{\pi}\tan^{-1}\frac{y_0}{x_0}$$

Since $\tan^{-1} x - \tan^{-1}(1/x) = \pi/2$ the total charge induced is

$$Q = -\lambda$$

which is, of course, also the sum of the charge per unit length of the three image charges.

(d) We have

$$\Phi = \frac{\lambda}{4\pi\epsilon_0} \ln \frac{r_2^2 r_3^2}{r_1^2 r_4^2}$$

Far from the origin,

$$\begin{aligned} r_1^2 &= \left[(x-x_0)^2 + (y-y_0)^2\right] \\ &= \left[x^2(1-\frac{x_0}{x})^2 + y^2(1-\frac{y_0}{y})^2\right] \\ &\approx \left[x^2(1-2\frac{x_0}{x}) + y^2(1-2\frac{y_0}{y}\right] \\ &= \left[x^2 - 2x_0 x + y^2 - 2y_0 y)\right] \\ &= (x^2+y^2)\left[1 - 2\frac{xx_0 + yy_0}{x^2+y^2}\right] \end{aligned}$$

Similarly,

$$\begin{aligned} r_2^2 &= (x^2+y^2)\left[1 - 2\frac{-xx_0 + yy_0}{x^2+y^2}\right] \\ r_3^2 &= (x^2+y^2)\left[1 - 2\frac{xx_0 - yy_0}{x^2+y^2}\right] \\ r_4^2 &= (x^2+y^2)\left[1 - 2\frac{-xx_0 - yy_0}{x^2+y^2}\right] \end{aligned}$$

Next,

$$\begin{aligned} r_1^2 r_4^2 &= (x^2+y^2)^2\left[1 - 4\frac{(xx_0 + yy_0)^2}{(x^2+y^2)^2}\right] \\ r_2^2 r_3^2 &= (x^2+y^2)^2\left[1 - 4\frac{(xx_0 - yy_0)^2}{(x^2+y^2)^2}\right] \end{aligned}$$

so

$$\Phi = \frac{\lambda}{4\pi\epsilon_0} \ln \left[\frac{1 - 4\frac{(xx_0-yy_0)^2}{(x^2+y^2)^2}}{1 - 4\frac{(xx_0+yy_0)^2}{(x^2+y^2)^2}}\right].$$

The (x^2+y^2) term in the denominator grows much more quickly than the (xx_0+yy_0) term, so in the asymptotic limit we can use $\ln(1+\epsilon) \approx \epsilon$ to find

$$\begin{aligned} \Phi &= \frac{\lambda}{4\pi\epsilon_0}\left[-4\frac{(xx_0-yy_0)^2}{(x^2+y^2)^2} + 4\frac{(xx_0+yy_0)^2}{(x^2+y^2)^2}\right] \\ &= \frac{\lambda}{4\pi\epsilon_0}\left[\frac{-4(x^2x_0^2 + y^2y_0^2 - 2xyx_0y_0) + 4(x^2x_0^2 + y^2y_0^2 + 2xyx_0y_0)}{(x^2+y^2)^2}\right] \end{aligned}$$

$$= \frac{\lambda}{4\pi\epsilon_0}\left[\frac{16xyx_0y_0}{(x^2+y^2)^2}\right]$$
$$= \frac{4\lambda}{\pi\epsilon_0}\frac{(xy)(x_0y_0)}{(x^2+y^2)^2}. \qquad \surd$$

Problem 2.4

A point charge is placed a distance $d > R$ from the center of an equally charged, isolated, conducting sphere of radius R.

(a) Inside of what distance from the surface of the sphere is the point charge attracted rather than repelled by the charged sphere?

(b) What is the limiting value of the force of attraction when the point charge is located a distance $a(= d-R)$ from the surface of the sphere, if $a \ll R$?

(c) What are the results for parts a and b if the charge on the sphere is twice (half) as large as the point charge, but still the same sign?

Let's call the point charge q. The charged, isolated sphere may be replaced by two image charges. One image charge, of charge $q_1 = -(R/d)q$ at radius $r_1 = R^2/d$, is needed to make the potential equal at all points on the sphere. The second image charge, of charge $q_2 = q - q_1$ at the center of the sphere, is necessary to recreate the effect of the additional charge on the sphere (the "additional" charge is the extra charge on the sphere left over after you subtract the surface charge density induced by the point charge q).

The force on the point charge is the sum of the forces from the two image charges:

$$F = \frac{1}{4\pi\epsilon_0}\left[\frac{qq_1}{\left[d-\frac{R^2}{d}\right]^2}+\frac{qq_2}{d^2}\right] \tag{4}$$
$$= \frac{q^2}{4\pi\epsilon_0}\left[\frac{-dR}{[d^2-R^2]^2}+\frac{d^2+dR}{d^4}\right] \tag{5}$$

As $d \to R$ the denominator of the first term vanishes, so that term wins, and the overall force is attractive. As $d \to \infty$, the denominator of both terms looks like d^4, so the dR terms in the numerator cancel and the overall force is repulsive.

(a) The crossover distance is found by equating the two bracketed terms in (5):

$$\begin{aligned}\frac{dR}{[d^2-R^2]^2} &= \frac{d^2+dR}{d^4}\\ d^4R &= (d+R)[d^2-R^2]^2\\ 0 &= d^5-2d^3R^2-2d^2R^3+dR^4+R^5\end{aligned}$$

I used GnuPlot to solve this one graphically. The root is d/R=1.6178.

(b) The idea here is to set $d = R + a = R(1 + a/R)$ and find the limit of (4) as $a \to 0$.

$$\begin{aligned}F &= \frac{q^2}{4\pi\epsilon_0}\left[\frac{-R^2(1+\frac{a}{R})}{\left[R^2(1+\frac{a}{R})^2-R^2\right]^2}+\frac{R^2\left[(1+\frac{a}{R})^2+(1+\frac{a}{R})\right]}{R^4(1+\frac{a}{R})^4}\right]\\ &\approx \frac{q^2}{4\pi\epsilon_0}\left[\frac{-R^2-aR}{4a^2R^2}+\frac{(2R+3a)(R-4a)}{R^4}\right]\end{aligned}$$

The second term in brackets approaches the constant $2/R^2$ as $a \to 0$. The first term becomes $-1/4a^2$. So we have

$$F \to -\frac{q^2}{16\pi\epsilon_0 a^2}.$$

Note that only the first image charge (the one required to make the sphere an equipotential) contributes to the force as $d \to a$. The second image charge, the one which represents the difference between the actual charge on the sphere and the charge induced by the first image, makes no contribution in this limit. That means that the limiting value of the force will be as above regardless of the charge on the sphere.

(c) If the charge on the sphere is twice the point charge, then $q_2 = 2q - q_1 = q(2 + R/d)$. Then (5) becomes

$$F = \frac{q^2}{4\pi\epsilon_0}\left[-\frac{dR}{[d^2-R^2]^2}+\frac{2d^2+dR}{d^4}\right]$$

and the relevant equation becomes

$$0 = 2d^5 - 4d^3R^2 - 2d^2R^3 + 2dR^4 + R^5.$$

Again I solved graphically to find $d/R = 1.43$. If the charge on the sphere is half the point charge, then

$$F = \frac{q^2}{4\pi\epsilon_0}\left[-\frac{dR}{[d^2-R^2]^2}+\frac{d^2+2dR}{2d^4}\right]$$

and the equation is

$$0 = d^5 - 2d^3R^2 - 4d^2R^3 + dR^4 + 2R^5.$$

The root of this one is d/R=1.88.

Problem 2.5

(a) Show that the work done to remove the charge q from a distance $r > a$ to infinity against the force, Eq. (2.6), of a grounded conducting sphere is

$$W = \frac{q^2 a}{8\pi\epsilon_0 (r^2 - a^2)}.$$

Relate this result to the electrostatic potential, Eq. (2.3), and the energy discussion of Section 1.11.

(b) Repeat the calculation of the work done to remove the charge q against the force, Eq. (2.9), of an isolated charged conducting sphere. Show that the work done is

$$W = \frac{1}{4\pi\epsilon_0}\left[\frac{q^2 a}{2(r^2 - a^2)} - \frac{q^2 a}{2r^2} - \frac{qQ}{r}\right].$$

Relate the work to the electrostatic potential, Eq. (2.8), and the energy discussion of Section 1.11.

(a) The force is

$$|F| = \frac{q^2 a}{4\pi\epsilon_0} \frac{1}{y^3(1 - a^2/y^2)^2}$$

directed radially inward. The work is

$$\begin{aligned} W &= -\int_r^\infty F dy & (6) \\ &= \frac{q^2 a}{4\pi\epsilon_0}\int_r^\infty \frac{dy}{y^3(1 - a^2/y^2)^2} \\ &= \frac{q^2 a}{4\pi\epsilon_0}\int_r^\infty \frac{y dy}{(y^2 - a^2)^2} \\ &= \frac{q^2 a}{4\pi\epsilon_0}\int_{r^2-a^2}^\infty \frac{du}{2u^2} \\ &= \frac{q^2 a}{4\pi\epsilon_0}\left| -\frac{1}{2u}\right|_{r^2-a^2}^\infty \\ &= \frac{q^2 a}{8\pi\epsilon_0 (r^2 - a^2)} & (7) \end{aligned}$$

To relate this to earlier results, note that the image charge $q' = -(a/r)q$ is located at radius $r' = a^2/r$. The potential energy between the point charge and its image is

$$\begin{aligned} PE &= \frac{1}{4\pi\epsilon_0}\left(\frac{qq'}{|r-r'|}\right) \\ &= \frac{1}{4\pi\epsilon_0}\left(\frac{-q^2 a}{r(r-a^2/r)}\right) \\ &= \frac{1}{4\pi\epsilon_0}\left(\frac{-q^2 a}{r^2-a^2}\right) \end{aligned} \tag{8}$$

Result (7) is only half of (8). This would seem to violate energy conservation. It would seem that we could start with the point charge at infinity and allow it to fall in to a distance r from the sphere, liberating a quantity of energy (8), which we could store in a battery or something. Then we could expend an energy equal to (7) to remove the charge back to infinity, at which point we would be back where we started, but we would still have half of the energy saved in the battery. It would seem that we could keep doing this over and over again, storing up as much energy in the battery as we pleased.

I think the problem is with equation (8). The traditional expression $q_1 q_2/4\pi\epsilon_0 r$ for the potential energy of two charges comes from calculating the work needed to bring one charge from infinity to a distance r from the other charge, and it is assumed that the other charge does not move and keeps a constant charge during the process. But in this case one of the charges is a fictitious image charge, and as the point charge q is brought in from infinity the image charge moves out from the center of the sphere, and its charge increases. So the simple expression doesn't work to calculate the potential energy of the configuration, and we should take (7) to be the correct result.

(b) In this case there are two image charges: one of the same charge and location as in part a, and another of charge $Q-q'$ at the origin. The work needed to remove the point charge q to infinity is the work needed to remove the point charge from its image charge, plus the work needed to remove the point charge from the extra charge at the origin. We calculated the first contribution above. The second contribution is

$$\begin{aligned} -\int_r^\infty \frac{q(Q-q')dy}{4\pi\epsilon_0 y^2} &= -\frac{1}{4\pi\epsilon_0}\int_r^\infty\left[\frac{qQ}{y^2}+\frac{q^2 a}{y^3}\right]dy \\ &= -\frac{1}{4\pi\epsilon_0}\left|-\frac{qQ}{y}-\frac{q^2 a}{2y^2}\right|_r^\infty \\ &= -\frac{1}{4\pi\epsilon_0}\left[\frac{qQ}{r}+\frac{q^2 a}{2r^2}\right] \end{aligned}$$

so the total work done is

$$W = \frac{1}{4\pi\epsilon_0}\left[\frac{q^2 a}{2(r^2-a^2)}-\frac{q^2 a}{2r^2}-\frac{qQ}{r}\right].$$

Review of Green's Functions

Some problems in this and other chapters use the Green's function technique. It's useful to review this technique, and also to establish my conventions since I define the Green's function a little differently than Jackson.

The whole technique is based on the divergence theorem. Suppose $\mathbf{A}(\mathbf{x})$ is a vector valued function defined at each point $\mathbf{x}$ within a volume V. Then

$$\int_V (\nabla \cdot \mathbf{A}(\mathbf{x}'))\, dV' = \oint_S \mathbf{A}(\mathbf{x}') \cdot \mathbf{dA}' \tag{9}$$

where S is the (closed) surface bounding the volume V. If we take $\mathbf{A}(\mathbf{x}) = \phi(\mathbf{x})\nabla\psi(\mathbf{x})$ where ϕ and ψ are scalar functions, (9) becomes

$$\int_V \left[(\nabla\phi(\mathbf{x}')) \cdot (\nabla\psi(\mathbf{x}')) + \phi(\mathbf{x}')\nabla^2\psi(\mathbf{x}')\right] dV' = \oint_S \phi(\mathbf{x}') \left.\frac{\partial\psi}{\partial n}\right|_{\mathbf{x}'} dA'$$

where $\partial\psi/\partial n$ is the dot product of $\vec{\nabla}\psi$ with the outward normal to the surface area element. If we write down this equation with ϕ and ψ switched and subtract the two, we come up with

$$\int_V \left[\phi\nabla^2\psi - \psi\nabla^2\phi\right] dV' = \oint_S \left[\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right] dA'. \tag{10}$$

This statement doesn't appear to be very useful, since it seems to require that we know ϕ over the whole volume to compute the left side, and both ϕ and $\partial\phi/\partial n$ on the boundary to compute the right side. However, suppose we could choose $\psi(\mathbf{x})$ in a clever way such that $\nabla^2\psi = \delta(\mathbf{x} - \mathbf{x}_0)$ for some point x_0 within the volume. (Since this ψ is a function of $\mathbf{x}$ which also depends on $\mathbf{x}_0$ as a parameter, we might write it as $\psi_{\mathbf{x}_0}(\mathbf{x})$.) Then we could use the sifting property of the delta function to find

$$\phi(\mathbf{x}_0) = \int_V \left[\psi_{\mathbf{x}_0}(\mathbf{x}')\nabla^2\phi(\mathbf{x}')\right] dV' + \oint_S \left[\phi(\mathbf{x}') \left.\frac{\partial\psi_{\mathbf{x}_0}}{\partial n}\right|_{\mathbf{x}'} - \psi_{\mathbf{x}_0}(\mathbf{x}') \left.\frac{\partial\phi}{\partial n}\right|_{\mathbf{x}'}\right] dA'.$$

If ϕ is the scalar potential of electrostatics, we know that $\nabla^2\psi(\mathbf{x}') = -\rho(\mathbf{x}')/\epsilon_0$, so we have

$$\phi(\mathbf{x}_0) = -\frac{1}{\epsilon_0}\int_V \psi_{\mathbf{x}_0}(\mathbf{x}')\rho(\mathbf{x}')dV' + \oint_S \left[\phi(\mathbf{x}') \left.\frac{\partial\psi_{\mathbf{x}_0}}{\partial n}\right|_{\mathbf{x}'} - \psi_{\mathbf{x}_0}(\mathbf{x}') \left.\frac{\partial\phi}{\partial n}\right|_{\mathbf{x}'}\right] dA'. \tag{11}$$

Equation (11) allows us to find the potential at an arbitrary point $\mathbf{x}_0$ as long as we know ρ within the volume and both ϕ and $\partial\phi/\partial n$ on the boundary. boundary. Usually we do know ρ within the volume, but we only know either ϕ **or** $\partial\phi/\partial n$ on the boundary. This lack of knowledge can be accommodated by choosing ψ such that either its value or its normal derivative vanishes on the boundary surface, so that the term which we can't evaluate drops out of the surface integral. More specifically,

- if we know ϕ but not $\partial\phi/\partial n$ on the boundary ("Dirichlet" boundary conditions), we choose ψ such that $\psi = 0$ on the boundary. Then

$$\phi(\mathbf{x}_0) = -\frac{1}{\epsilon_0}\int_V \psi_{\mathbf{x}_0}(\mathbf{x}')\rho(\mathbf{x}')dV' + \oint_S \phi(\mathbf{x}')\left.\frac{\partial\psi_{\mathbf{x}_0}}{\partial n}\right|_{\mathbf{x}'} dA'. \tag{12}$$

- if we know $\partial\phi/\partial n$ but not ϕ on the boundary ("Neumann" boundary conditions), we choose ψ such that $\partial\psi/\partial n = 0$ on the boundary. Then

$$\phi(\mathbf{x}_0) = -\frac{1}{\epsilon_0}\int_V \psi_{\mathbf{x}_0}(\mathbf{x}')\rho(\mathbf{x}')dV' + \oint_S \phi_{\mathbf{x}_0}(\mathbf{x}')\left.\frac{\partial\phi}{\partial n}\right|_{\mathbf{x}'} dA'. \tag{13}$$

Again, in both cases the function $\psi_{\mathbf{x}_0}(\mathbf{x})$ has the property that

$$\nabla^2\psi_{\mathbf{x}_0}(\mathbf{x}) = \delta(\mathbf{x} - \mathbf{x}_0).$$

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December 8, 1999

Chapter 2: Problems 11-20

Problem 2.11

A line charge with linear charge density τ is placed parallel to, and a distance R away from, the axis of a conducting cylinder of radius b held at fixed voltage such that the potential vanishes at infinity. Find

(a) the magnitude and position of the image charge(s);

(b) the potential at any point (expressed in polar coordinates with the origin at the axis of the cylinder and the direction from the origin to the line charge as the x axis), including the asymptotic form far from the cylinder;

(c) the induced surface-charge density, and plot it as a function of angle for R/b=2,4 in units of $\tau/2\pi b$;

(d) the force on the charge.

(a) Drawing an analogy to the similar problem of the point charge outside the conducting sphere, we might expect that the potential on the cylinder can be made constant by placing an image charge within the cylinder on the line conducting the line charge with the center of the cylinder, i.e. on the x axis. Suppose we put the image charge a distance $R' < b$ from the center of the cylinder and give it a charge density $-\tau$. Using the expression quoted in Problem 2.3 for the potential of a line charge, the potential at a point $\mathbf{x}$ due to the line charge and its image is

$$\Phi(\mathbf{x}) = \frac{\tau}{4\pi\epsilon_0}\ln\frac{R^2}{|\mathbf{x}-R\hat{\mathbf{i}}|^2} - \frac{\tau}{4\pi\epsilon_0}\ln\frac{R^2}{|\mathbf{x}-R'\hat{\mathbf{i}}|^2}$$

$$= \quad \frac{\tau}{4\pi\epsilon_0} \ln \frac{|\mathbf{x} - R'\hat{\mathbf{i}}|^2}{|\mathbf{x} - R\hat{\mathbf{i}}|^2}.$$

We want to choose R' such that the potential is constant when $\mathbf{x}$ is on the cylinder surface. This requires that the argument of the logarithm be equal to some constant γ at those points:

$$\frac{|\mathbf{x} - R'\hat{\mathbf{i}}|^2}{|\mathbf{x} - R\hat{\mathbf{i}}|^2} = \gamma$$

or

$$b^2 + R'^2 - 2R'b\cos\phi = \gamma b^2 + \gamma R^2 - 2\gamma Rb\cos\phi.$$

For this to be true everywhere on the cylinder, the ϕ term must drop out, which requires $R' = \gamma R$. We can then rearrange the remaining terms to find

$$R' = \frac{b^2}{R}.$$

This is also analogous to the point-charge-and-sphere problem, but there are differences: in this case the image charge has the same magnitude as the original line charge, and the potential on the cylinder is constant but not zero.

(b) At a point (ρ, ϕ), we have

$$\Phi = \frac{\tau}{4\pi\epsilon_0} \ln \frac{\rho^2 + R'^2 - 2\rho R'\cos\phi}{\rho^2 + R^2 - 2\rho R\cos\phi}.$$

For large ρ, this becomes

$$\Phi \to \frac{\tau}{4\pi\epsilon_0} \ln \frac{1 - 2\frac{R'}{\rho}\cos\phi}{1 - 2\frac{R}{\rho}\cos\phi}.$$

Using $\ln(1-x) = -(x + x^2/2 + \cdots)$, we have

$$\begin{aligned}
\Phi \quad &\to \quad \frac{\tau}{4\pi\epsilon_0} \frac{2(R - R')\cos\phi}{\rho} \\
&= \quad \frac{\tau}{2\pi\epsilon_0} \frac{R(1 - b^2/R^2)\cos\phi}{\rho}
\end{aligned}$$

(c)

$$\begin{aligned}
\sigma \quad &= \quad -\epsilon_0 \left.\frac{\partial\Phi}{\partial\rho}\right|_{r=b} \\
&= \quad -\frac{\tau}{4\pi}\left[\frac{2b - 2R'\cos\phi}{b^2 + R'^2 - 2bR'\cos\phi} - \frac{2b - 2R\cos\phi}{b^2 + R^2 - 2bR\cos\phi}\right] \\
&= \quad -\frac{\tau}{2\pi}\left[\frac{b - \frac{b^2}{R}\cos\phi}{b^2 + \frac{b^4}{R^2} - 2\frac{b^3}{R}\cos\phi} - \frac{b - R\cos\phi}{b^2 + R^2 - 2bR\cos\phi}\right]
\end{aligned}$$

Multiplying the first term by R^2/b^2 on top and bottom yields

$$\begin{aligned}\sigma &= -\frac{\tau}{2\pi}\left[\frac{\frac{R^2}{b}-b}{R^2+b^2-2bR\cos\phi}\right] \\ &= -\frac{\tau}{2\pi b}\left[\frac{R^2-b^2}{R^2+b^2-2bR\cos\phi}\right]\end{aligned}$$

(d) To find the force on the charge, we note that the potential of the image charge is

$$\Phi(\mathbf{x}) = -\frac{\tau}{4\pi\epsilon_0}\ln\frac{C^2}{|\mathbf{x}-R'\,\hat{\mathbf{i}}|^2}.$$

with C some constant. We can differentiate this to find the electric field due to the image charge:

$$\begin{aligned}\mathbf{E}(\mathbf{x}) &= -\nabla\Phi(\mathbf{x}) = -\frac{\tau}{4\pi\epsilon_0}\nabla\ln|\mathbf{x}-R'\,\hat{\mathbf{i}}|^2 \\ &= -\frac{\tau}{4\pi\epsilon_0}\frac{2(\mathbf{x}-R'\,\hat{\mathbf{i}})}{|\mathbf{x}-R'\hat{\mathbf{i}}|^2}.\end{aligned}$$

The original line charge is at $x = R, y = 0$, and the field there is

$$\mathbf{E} = -\frac{\tau}{2\pi\epsilon_0}\frac{1}{R-R'}\hat{\mathbf{i}} = -\frac{\tau}{2\pi\epsilon_0}\frac{R}{R^2-b^2}\,\hat{\mathbf{i}}.$$

The force per unit width on the line charge is

$$F = \tau E = -\frac{\tau^2}{2\pi\epsilon_0}\frac{R}{R^2-b^2}$$

tending to pull the original charge in toward the cylinder.

Problem 2.12

Starting with the series solution (2.71) for the two-dimensional potential problem with the potential specified on the surface of a cylinder of radius b, evaluate the coefficients formally, substitute them into the series, and sum it to obtain the potential *inside* the cylinder in the form of Poisson's integral:

$$\Phi(\rho,\phi) = \frac{1}{2\pi}\int_0^{2\pi}\Phi(b,\phi')\frac{b^2-\rho^2}{b^2+\rho^2-2b\rho\cos(\phi'-\phi)}d\phi'$$

What modification is necessary if the potential is desired in the region of space bounded by the cylinder and infinity?

Referring to equation (2.71), we know the b_n are all zero, because the ln term and the negative powers of ρ are singular at the origin. We are left with

$$\Phi(\rho, \phi) = a_0 + \sum_{n=1}^{\infty} \rho^n \left\{ a_n \sin(n\phi) + b_n \cos(n\phi) \right\}. \tag{1}$$

Multiplying both sides successively by 1, $\sin n'\phi$, and $\cos n'\phi$ and integrating at $\rho = b$ gives

$$a_0 = \frac{1}{2\pi} \int_0^{2\pi} \Phi(b, \phi) d\phi \tag{2}$$

$$a_n = \frac{1}{\pi b^n} \int_0^{2\pi} \Phi(b, \phi) \sin(n\phi) d\phi \tag{3}$$

$$b_n = \frac{1}{\pi b^n} \int_0^{2\pi} \Phi(b, \phi) \cos(n\phi) d\phi. \tag{4}$$

Plugging back into (1), we find

$$\begin{aligned} \Phi(\rho, \phi) &= \frac{1}{\pi} \int_0^{2\pi} \Phi(b, \phi') \left\{ \frac{1}{2} + \sum_{n=1}^{\infty} \left(\frac{\rho}{b} \right)^n \left[\sin(n\phi) \sin(n\phi') + \cos(n\phi) \cos(n\phi') \right] \right\} d\phi' \\ &= \frac{1}{\pi} \int_0^{2\pi} \Phi(b, \phi') \left\{ \frac{1}{2} + \sum_{n=1}^{\infty} \left(\frac{\rho}{b} \right)^n \cos n(\phi - \phi') \right\}. \end{aligned} \tag{5}$$

The bracketed term can be expressed in closed form. For simplicity define $x = (\rho/b)$ and $\alpha = (\phi - \phi')$. Then

$$\begin{aligned} \frac{1}{2} + \sum_{n=1}^{\infty} x^n \cos(n\alpha) &= \frac{1}{2} + \frac{1}{2} \sum_{n=1}^{\infty} \left[x^n e^{in\alpha} + x^n e^{-in\alpha} \right] \\ &= \frac{1}{2} + \frac{1}{2} \left[\frac{1}{1 - xe^{i\alpha}} + \frac{1}{1 - xe^{-i\alpha}} - 2 \right] \\ &= \frac{1}{2} + \frac{1}{2} \left[\frac{1 - xe^{-i\alpha} - xe^{i\alpha} + 1}{1 - xe^{i\alpha} - xe^{-i\alpha} + x^2} - 2 \right] \\ &= \frac{1}{2} + \left[\frac{1 - x \cos\alpha}{1 + x^2 - 2x\cos\alpha} - 1 \right] \\ &= \frac{1}{2} + \frac{x\cos\alpha - x^2}{1 + x^2 - 2x\cos\alpha} \\ &= \frac{1}{2} \left[\frac{1 - x^2}{1 + x^2 - 2x\cos\alpha} \right]. \end{aligned}$$

Plugging this back into (5) gives the advertised result.

Problem 2.13

(a) Two halves of a long hollow conducting cylinder of inner radius b are separated by small lengthwise gaps on each side, and are kept at different potentials V_1 and V_2. Show that the potential inside is given by

$$\Phi(\rho,\phi) = \frac{V_1+V_2}{2} + \frac{V_1-V_2}{\pi}\tan^{-1}\left(\frac{2b\rho}{b^2-\rho^2}\cos\phi\right)$$

where ϕ is measured from a plane perpendicular to the plane through the gap.
(b) Calculate the surface-charge density on each half of the cylinder.

This problem is just like the previous one. Since we are looking for an expression for the potential within the cylinder, the correct expansion is (1) with expansion coefficients given by (2), (3) and (4):

$$\begin{aligned}
a_0 &= \frac{1}{2\pi}\int_0^{2\pi}\Phi(b,\phi)d\phi \\
&= \frac{1}{2\pi}\left[V_1\int_0^{\pi}d\phi + V_2\int_{\pi}^{2\pi}d\phi\right] \\
&= \frac{V_1+V_2}{2} \\
a_n &= \frac{1}{\pi b^n}\left[V_1\int_0^{\pi}\sin(n\phi)d\phi + V_2\int_{\pi}^{2\pi}\sin(n\phi)d\phi\right] \\
&= -\frac{1}{n\pi b^n}\left[V_1\left|\cos n\phi\right|_0^{\pi} + V_2\left|\cos n\phi\right|_{\pi}^{2\pi}\right] \\
&= -\frac{1}{n\pi b^n}\left[V_1(\cos n\pi - 1) + V_2(1-\cos n\pi)\right] \\
&= \left\{\begin{array}{ll} 0 & \text{, n even} \\ 2(V_1-V_2)/(n\pi b^n) & \text{, n odd}\end{array}\right\} \\
b_n &= \frac{1}{\pi b^n}\left[V_1\int_0^{\pi}\cos(n\phi)d\phi + V_2\int_{\pi}^{2\pi}\cos(n\phi)d\phi\right] \\
&= \frac{1}{n\pi b^n}\left[V_1\left|\sin n\phi\right|_0^{\pi} + V_2\left|\sin n\phi\right|_{\pi}^{2\pi}\right] \\
&= 0.
\end{aligned}$$

With these coefficients, the potential expansion becomes

$$\Phi(\rho,\phi) = \frac{V_1+V_2}{2} + \frac{2(V_1-V_2)}{\pi}\sum_{\text{n odd}}\frac{1}{n}\left(\frac{\rho}{b}\right)^n\sin n\phi. \tag{6}$$

Here we need an auxiliary result:

$$\begin{aligned}\sum_{\text{n odd}} \frac{1}{n} x^n \sin n\phi &= \frac{1}{2i} \sum_{\text{n odd}} \frac{1}{n} (iy)^n [e^{in\pi} - e^{-in\phi}] \qquad (x = iy) \\ &= \frac{1}{2} \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \left[(ye^{i\phi})^{2n+1} - (ye^{-i\phi})^{2n+1} \right] \\ &= \frac{1}{2} \left[\tan^{-1}(ye^{i\phi}) - \tan^{-1}(ye^{-i\phi}) \right] \qquad (7)\end{aligned}$$

where in the last line we just identified the Taylor series for the inverse tangent function. Next we need an identity:

$$\tan^{-1}\gamma_1 - \tan^{-1}\gamma_2 = \tan^{-1}\left(\frac{\gamma_1 - \gamma_2}{1 + \gamma_1\gamma_2}\right).$$

(I derived this one by drawing some triangles and doing some algebra.) With this, (7) becomes

$$\begin{aligned}\sum_{\text{n odd}} \frac{1}{n} x^n \sin n\phi &= \frac{1}{2} \tan^{-1}\left(\frac{2iy \sin\phi}{1 + y^2}\right) \\ &= \frac{1}{2} \tan^{-1}\left(\frac{2x \sin\phi}{1 - x^2}\right).\end{aligned}$$

Using this in (6) with $x = \rho/b$ gives

$$\Phi(\rho, b) = \frac{V_1 + V_2}{\pi} + \frac{V_1 - V_2}{\pi} \tan^{-1}\left(\frac{2\rho b \sin\phi}{b^2 - \rho^2}\right).$$

(Evidently, Jackson and I defined the angle ϕ differently).

Problem 2.15

(a) Show that the Green function $G(x, y; x', y')$ appropriate for Dirichlet boundary conditions for a square two-dimensional region, $0 \le x \le 1, 0 \le y \le 1$, has an expansion

$$G(x, y; x', y') = 2\sum_{n=1}^{\infty} g_n(y, y') \sin(n\pi x) \sin(n\pi x')$$

where $g_n(y, y')$ satisfies

$$\left(\frac{\partial^2}{\partial y^2} - n^2\pi^2\right) g_n(y, y') = \delta(y' - y) \qquad \text{and} \qquad g_n(y, 0) = g_n(y, 1) = 0.$$

(b) Taking for $g_n(y, y')$ appropriate linear combinations of $\sinh(n\pi y')$ and $\cosh(n\pi y')$ in the two regions $y' < y$ and $y' > y$, in accord with the boundary conditions and the discontinuity in slope required by the source delta function, show that the explicit form of G is

$$G(x, y; x', y') =$$
$$-2\sum_{n=1}^{\infty} \frac{1}{n\pi \sinh(n\pi)} \sin(n\pi x) \sin(n\pi x') \sinh(n\pi y_<) \sinh[n\pi(1 - y_>)]$$

where $y_<$ ($y_>$) is the smaller (larger) of y and y'.

(I have taken out a factor -4π from the expressions for g_n and G, in accordance with my convention for Green's functions; see the Green's functions review above.)

(a) To use as a Green's function in a Dirichlet boundary value problem G must satisfy two conditions. The first is that G vanish on the boundary of the region of interest. The suggested expansion of G clearly satisfies this. First, $\sin(n\pi x')$ is 0 when x' is 0 or 1. Second, $g(y, y')$ vanishes when y' is 0 or 1. So $G(x, y; x', y')$ vanishes for points (x', y') on the boundary.

The second condition on G is

$$\nabla^2 G = \left(\frac{\partial^2}{\partial x'^2} + \frac{\partial^2}{\partial y'^2}\right) G = \delta(x - x')\,\delta(y - y'). \tag{8}$$

With the suggested expansion, we have

$$\begin{aligned}
\frac{\partial^2}{\partial x'^2} G &= 2\sum_{n=1}^{\infty} g_n(y, y') \sin(n\pi x) \left[-n^2\pi^2 \sin(n\pi x')\right] \\
\frac{\partial^2}{\partial y'^2} G &= 2\sum_{n=1}^{\infty} \frac{\partial'^2}{\partial y^2} g_n(y, y') \sin(n\pi x) \sin(n\pi x')
\end{aligned}$$

We can add these together and use the differential equation satisfied by g_n to find

$$\begin{aligned}\nabla^2 G &= \delta(y-y') \cdot 2\sum_{n=1}^{\infty}\sin(n\pi x)\sin(n\pi x') \\ &= \delta(y-y') \cdot \delta(x-x')\end{aligned}$$

since the infinite sum is just a well-known representation of the δ function.

(b) The suggestion is to take

$$g_n(y,y') = \begin{cases} A_{n1}\sinh(n\pi y') + B_{n1}\cosh(n\pi y'), & y' < y; \\ A_{n2}\sinh(n\pi y') + B_{n2}\cosh(n\pi y'), & y' > y. \end{cases} \quad (9)$$

The idea to use hyperbolic sines and cosines comes from the fact that $\sinh(n\pi y)$ and $\cosh(n\pi y)$ satisfy a homogeneous version of the differential equation for g_n (i.e. satisfy that differential equation with the δ function replaced by zero). Thus g_n as defined in (9) satisfies its differential equation (at all points except $y = y'$) for *any* choice of the As and Bs. This leaves us free to choose these coefficients as required to satisfy the boundary conditions and the differential equation at $y = y'$.

First let's consider the boundary conditions. Since y is somewhere between 0 and 1, the condition that g_n vanish for $y' = 0$ is only relevant to the top line of (9), where it requires taking $B_{n1} = 0$ but leaves A_{n1} undetermined for now. The condition that g_n vanish for $y' = 1$ only affects the lower line of (9), where it requires that

$$\begin{aligned} 0 &= A_{n2}\sinh(n\pi) + B_{n2}\cosh(n\pi) \\ &= (A_{n2}+B_{n2})e^{n\pi} + (-A_{n2}+B_{n2})e^{-n\pi} \end{aligned} \quad (10)$$

One way to make this work is to take

$$A_{n2} + B_{n2} = -e^{-n\pi} \quad \text{and} \quad -A_{n2} + B_{n2} = e^{n\pi}.$$

Then

$$\begin{aligned} B_{n2} = e^{n\pi} + A_{n2} \quad &\rightarrow \quad 2A_{n2} = -e^{n\pi} - e^{-n\pi} \\ \text{so } A_{n2} = -\cosh(n\pi) \quad &\text{and} \quad B_{n2} = \sinh(n\pi). \end{aligned}$$

With this choice of coefficients, the lower line in (9) becomes

$$g_n(y,y') = -\cosh(n\pi)\sinh(n\pi y') + \sinh(n\pi)\cosh(n\pi y') = \sinh[n\pi(1-y')] \quad (11)$$

for $(y' > y)$. Actually, we haven't completely determined A_{n2} and B_{n2}; we could multiply (11) by an arbitrary constant γ_n and (10) would still be satisfied.

Next we need to make sure that the two halves of (9) match up at $y' = y$:

$$A_{n1}\sinh(n\pi y) = \gamma_n \sinh[n\pi(1-y)]. \quad (12)$$

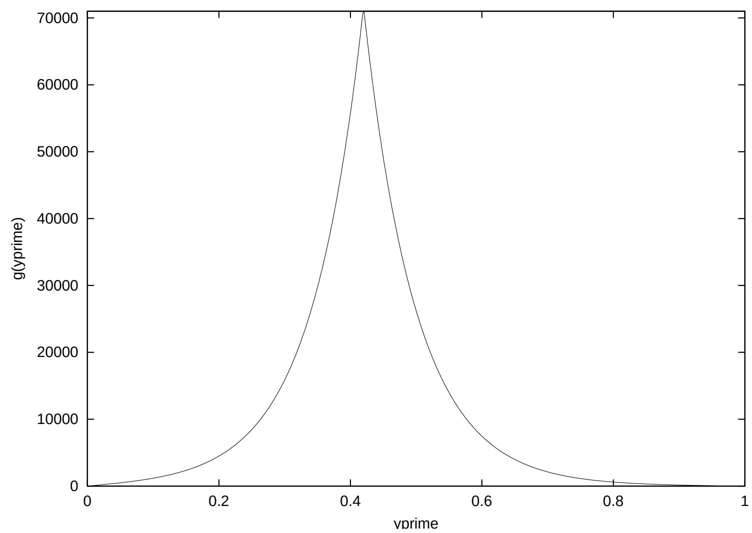

Figure 1: $g_n(y, y')$ from Problem 2.15 with n=5, y=.41

This obviously happens when

$$A_{n1} = \beta_n \sinh[n\pi(1-y)] \quad \text{and} \quad \gamma_n = \beta_n \sinh(n\pi y)$$

where β_n is any constant. In other words, we have

$$\begin{aligned} g_n(y, y') &= \begin{cases} \beta_n \sinh[n\pi(1-y)] \sinh(n\pi y'), & y' < y; \\ \beta_n \sinh[n\pi(1-y')] \sinh(n\pi y), & y' > y. \end{cases} \\ &= \beta_n \sinh[n\pi(1-y_>)] \sinh(n\pi y_<) \end{aligned} \tag{13}$$

with $y_<$ and $y_>$ defined as in the problem. Figure 1 shows a graph of this function $n = 5, y = .41$.

The final step is to choose the normalization constant β_n such that g_n satisfies its differential equation:

$$\left(\frac{\partial^2}{\partial^2 y'^2} - n^2\pi^2 \right) g_n(y, y') = \delta(y - y'). \tag{14}$$

To say that the left-hand side "equals" the delta function requires two things:

- that the left-hand side vanish at all points $y' \neq y$, and
- that its integral over any interval (y_1, y_2) equal 1 if the interval contains the point $y' = y$, and vanish otherwise.

The first condition is clearly satisfied regardless of the choice of β_n. The second condition may be satisfied by making g_n continuous, which we have already done, but giving its first derivative a finite jump of unit magnitude at $y' = y$:

$$\frac{\partial}{\partial y'} g_n(y,y')\Big|_{y'=y^-}^{y'=y^+} = 1.$$

Differentiating (13), we find this condition to require

$$n\pi\beta_n \left[-\cosh[n\pi(1-y)]\sinh(n\pi y) - \sinh[n\pi(1-y)]\cosh(n\pi y)\right] = -n\pi\beta_n \sinh(n\pi) = 1$$

so (14) is satisfied if

$$\beta_n = -\frac{1}{n\pi\sinh(n\pi)}.$$

Then (13) is

$$g_n(y,y') = -\frac{\sinh[n\pi(1-y_>)]\sinh(n\pi y_<)}{n\pi\sinh(n\pi)}$$

and the composite Green's function is

$$\begin{aligned} G(x,y;x',y') &= 2\sum_{n=1}^{\infty} g_n(y,y')\sin(n\pi x)\sin(n\pi x') \\ &= -2\sum_{n=1}^{\infty} \frac{\sinh[n\pi(1-y_>)]\sinh(n\pi y_<)\sin(n\pi x)\sin(n\pi x')}{n\pi\sinh(n\pi)} \end{aligned} \quad (15)$$

Problem 2.16

A two-dimensional potential exists on a unit square area ($0 \le x \le 1, 0 \le y \le 1$) bounded by "surfaces" held at zero potential. Over the entire square there is a uniform charge density of unit strength (per unit length in z). Using the Green function of Problem 2.15, show that the solution can be written as

$$\Phi(x,y) = \frac{4}{\pi^3\epsilon_0}\sum_{m=0}^{\infty}\frac{\sin[(2m+1)\pi x]}{(2m+1)^3}\left\{1 - \frac{\cosh[(2m+1)\pi(y-(1/2))]}{\cosh[(2m+1)\pi/2]}\right\}.$$

Referring to my Green's functions review above, the potential at a point $\mathbf{x}_0$ within the square is given by

$$\Phi(\mathbf{x}_0) = -\frac{1}{\epsilon_0}\int_V G(\mathbf{x}_0;\mathbf{x}')\rho(\mathbf{x}')dV' + \oint_S \left[\Phi(\mathbf{x}')\left.\frac{\partial G}{\partial n}\right|_{\mathbf{x}'} - G(\mathbf{x}_0;\mathbf{x}')\left.\frac{\partial \Phi}{\partial n}\right|_{\mathbf{x}'}\right] dA'. \quad (16)$$

In this case the surface integral vanishes, because we're given that Φ vanishes on the boundary, and G vanishes there by construction. We're also given that

$\rho(\mathbf{x}')dV' = dx'dy'$ throughout the entire volume. Then we can plug in (15) to find

$$\Phi(\mathbf{x}_0) = \frac{2}{\pi\epsilon_0}\sum_{n=1}^{\infty}\frac{1}{n\sinh(n\pi)}\int_0^1\int_0^1 \sinh[n\pi(1-y_>)]\sinh(n\pi y_<)\sin(n\pi x_0)\sin(n\pi x')dx'dy'. \tag{17}$$

The integrals can be done separately. The x integral is

$$\begin{aligned}\sin(n\pi x_0)\int_0^1 \sin(n\pi x')dx' &= -\frac{\sin(n\pi x_0)}{n\pi}[\cos(n\pi)-1] \\ &= \begin{cases} (2\sin(n\pi x_0))/n\pi & \text{, n odd} \\ 0 & \text{, n even}\end{cases}\end{aligned} \tag{18}$$

The y integral is

$$\begin{aligned} &\sinh[n\pi(1-y_0)]\int_0^{y_0}\sinh(n\pi y')dy' + \sinh(n\pi y_0)\int_{y_0}^1 \sinh[n\pi(1-y')]dy' \\ =\ & \frac{1}{n\pi}\left\{\sinh[n\pi(1-y_0)]\cdot\Big|\cosh(n\pi y')\Big|_0^{y_0} - \sinh[n\pi y_0]\cdot\Big|\cosh[n\pi(1-y')]\Big|_{y_0}^1\right\} \\ =\ & \frac{1}{n\pi}\{\sinh[n\pi(1-y_0)]\cosh(n\pi y_0) + \sinh(n\pi y_0)\cosh[n\pi(1-y_0)] - \sinh(n\pi y_0) - \sinh[n\pi(1-y_0)]\} \\ =\ & \frac{1}{n\pi}\{\sinh[n\pi] - \sinh[n\pi(1-y_0)] - \sinh(n\pi y_0)\}. \end{aligned} \tag{19}$$

Inserting (18) and (19) in (17), we have

$$\Phi(\mathbf{x}_0) = \frac{4}{\pi^3\epsilon_0}\sum_{\text{n odd}}\frac{\sin(n\pi x_0)}{n^3}\left\{1 - \frac{\sinh[n\pi(1-y_0)] + \sinh(n\pi y_0)}{\sinh(n\pi)}\right\}.$$

The thing in brackets is equal to what Jackson has, but this is tedious to show so I'll skip the proof.

Problem 2.17

(a) Construct the free-space Green function $G(x,y;x',y')$ for two-dimensional electrostatics by integrating $1/R$ with respect to $z'-z$ between the limits $\pm Z$, where Z is taken to be very large. Show that apart from an inessential constant, the Green function can be written alternately as

$$\begin{aligned} G(x,y;x',y') &= -\ln[(x-x')^2+(y-y')^2] \\ &= -\ln[\rho^2+\rho'^2-2\rho\rho'\cos(\phi-\phi')]. \end{aligned}$$

(b) Show explicitly by separation of variables in polar coordinates that the Green function can be expressed as a Fourier series in the azimuthal coordinate,

$$G = \frac{1}{2\pi}\sum_{-\infty}^{\infty} e^{im(\phi-\phi')} g_m(\rho,\rho')$$

where the radial Green functions satisfy

$$\frac{1}{\rho'}\frac{\partial}{\partial\rho'}\left(\rho'\frac{\partial g_m}{\partial\rho'}\right) - \frac{m^2}{\rho'^2}g_m = \frac{\delta(\rho-\rho')}{\rho}.$$

Note that $g_m(\rho,\rho')$ for fixed ρ is a different linear combination of the solutions of the homogeneous radial equation (2.68) for $\rho' < \rho$ and for $\rho' > \rho$, with a discontinuity of slope at $\rho' = \rho$ determined by the source delta function.

(c) Complete the solution and show that the free-space Green function has the expansion

$$G(\rho,\phi;\rho',\phi') = \frac{1}{4\pi}\ln(\rho_>^2) - \frac{1}{2\pi}\sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{\rho_<}{\rho_>}\right)^m \cdot \cos[m(\phi-\phi')]$$

where $\rho_<(\rho_>)$ is the smaller (larger) of ρ and ρ'.

(As in Problem 2.15, I modified the text of the problem to match with my convention for Green's functions.)

(a)

$$\begin{aligned} R &= [(x-x')^2+(y-y')^2+(z-z')^2]^{1/2} \\ &\equiv [a^2+u^2]^{1/2} \quad , \quad a = [(x-x')^2+(y-y')^2]^{1/2} \quad , \quad u = (z-z'). \end{aligned}$$

Integrating,

$$\int_{-Z}^{Z}\frac{du}{[a^2+u^2]^{1/2}} = \left|\ln\left[(a^2+u^2)^{1/2}+u\right]\right|_{-Z}^{+Z}$$

$$
\begin{aligned}
&= \ln \frac{(Z^2+a^2)^{1/2}+Z}{(Z^2+a^2)^{1/2}-Z} \\
&= \ln \frac{(1+(a^2/Z^2))^{1/2}+1}{(1+(a^2/Z^2))^{1/2}-1} \\
&\approx \ln \frac{2+\frac{a^2}{2Z^2}}{\frac{a^2}{2Z^2}} \\
&= \ln \frac{4Z^2+a^2}{a^2} \\
&= \ln[4Z^2+a^2] - \ln a^2.
\end{aligned}
$$

Since Z is much bigger than a, the first term is essentially independent of a and is the 'nonessential constant' Jackson is talking about. The remaining term is the 2D Green's function:

$$
\begin{aligned}
G = -\ln a^2 &= -\ln[(x-x')^2+(y-y')^2] \text{ in rectangular coordinates} \\
&= -\ln[\rho^2+\rho'^2-2\rho\rho'\cos(\phi-\phi')] \text{ in cylindrical coordinates.}
\end{aligned}
$$

(b) The 2d Green's function is defined by

$$\int \nabla^2 G(\rho,\phi;\rho',\phi')\rho' d\rho' d\phi' = 1$$

but $\nabla^2 G = 0$ at points other than (ρ,ϕ). These conditions are met if

$$\nabla^2 G(\rho,\phi;\rho',\phi') = \frac{1}{\rho'}\delta(\rho-\rho')\delta(\phi-\phi'). \tag{20}$$

You need the ρ' on the bottom there to cancel out the ρ' in the area element in the integral. The Laplacian in two-dimensional cylindrical coordinates is

$$\nabla^2 = \frac{1}{\rho'}\frac{\partial}{\partial\rho'}\left(\rho'\frac{\partial}{\partial\rho'}\right) - \frac{1}{\rho'^2}\frac{\partial}{\partial\phi'^2}.$$

Applying this to the suggested expansion for G gives

$$\nabla^2 G(\rho,\phi;\rho',\phi') = \frac{1}{2\pi}\sum_{-\infty}^{\infty}\left\{\frac{1}{\rho'}\frac{\partial}{\partial\rho'}\left(\rho'\frac{\partial g_m}{\partial\rho'}\right) - \frac{m^2}{\rho'^2}g_m\right\}e^{im(\phi-\phi')}.$$

If g_m satisfies its differential equation as specified in the problem, the term in brackets equals $\delta(\rho-\rho')/\rho'$ for all m and may be removed from the sum, leaving

$$
\begin{aligned}
\nabla^2 G(\rho,\phi;\rho',\phi') &= \left(\frac{\delta(\rho-\rho')}{\rho'}\right)\cdot\frac{1}{2\pi}\sum_{-\infty}^{\infty}e^{im(\phi-\phi')} \\
&= \left(\frac{\delta(\rho-\rho')}{\rho'}\right)\delta(\phi-\phi').
\end{aligned}
$$

(c) As in Problem 2.15, we'll construct the functions g_m by finding solutions of the homogenous radial differential equation in the two regions and piecing them together at $\rho = \rho'$ such that the function is continuous but its derivative has a finite jump of magnitude $1/\rho$.

For $m \geq 1$, the solution to the homogenous equation

$$\left\{ \frac{1}{\rho'} \frac{\partial}{\partial \rho'} \left(\rho' \frac{\partial}{\partial \rho'} \right) - \frac{m^2}{\rho'^2} \right\} f(\rho') = 0$$

is

$$f(\rho') = A_m \rho'^m + B_m \rho'^{-m}.$$

Thus we take

$$g_m = \begin{cases} A_{1m}\rho'^m + B_{1m}\rho'^{-m} & , \quad \rho' < \rho \\ A_{2m}\rho'^m + B_{2m}\rho'^{-m} & , \quad \rho' > \rho. \end{cases}$$

In order that the first solution be finite at the origin, and the second solution be finite at infinity, we have to take $B_{1m} = A_{2m} = 0$. Then the condition that the two solutions match at $\rho = \rho'$ is

$$A_{1m}\rho^m = B_{2m}\rho^{-m}$$

which requires

$$A_{1m} = \gamma_m \rho^{-m} \qquad B_{2m} = \rho^m \gamma_m$$

for some constant γ_m. Now we have

$$g_m = \begin{cases} \gamma_m \left(\frac{\rho'}{\rho}\right)^m & , \quad \rho' < \rho \\ \gamma_m \left(\frac{\rho}{\rho'}\right)^m & , \quad \rho' > \rho \end{cases}$$

The finite-derivative step condition is

$$\left.\frac{dg_m}{d\rho'}\right|_{\rho'=\rho+} - \left.\frac{dg_m}{d\rho'}\right|_{\rho'=\rho-} = \frac{1}{\rho}$$

or

$$-m\gamma_m \left(\frac{1}{\rho} + \frac{1}{\rho} \right) = \frac{1}{\rho}$$

so

$$\gamma_m = -\frac{1}{2m}.$$

Then

$$\begin{aligned} g_m &= \begin{cases} -\frac{1}{2m}\left(\frac{\rho'}{\rho}\right)^m & , \quad \rho' < \rho \\ -\frac{1}{2m}\left(\frac{\rho}{\rho'}\right)^m & , \quad \rho' > \rho \end{cases} \\ &= -\frac{1}{2m}\left(\frac{\rho_<}{\rho_>}\right)^m. \end{aligned}$$

Plugging this back into the expansion gives

$$\begin{aligned} G &= -\frac{1}{4\pi}\sum_{-\infty}^{\infty}\frac{1}{m}\left(\frac{\rho_<}{\rho_>}\right)^m e^{im(\phi-\phi')} \\ &= -\frac{1}{2\pi}\sum_{1}^{\infty}\frac{1}{m}\left(\frac{\rho_<}{\rho_>}\right)^m \cos[m(\phi-\phi')]. \end{aligned}$$

Jackson seems to be adding a ln term to this, which comes from the $m = 0$ solution of the radial equation, but I have left it out because it doesn't vanish as $\rho' \to \infty$.

Problem 2.18

(a) By finding appropriate solutions of the radial equation in part b of Problem 2.17, find the Green function for the interior Dirichlet problem of a cylinder of radius b [$g_m(\rho, \rho' = b) = 0$. See (1.40)]. First find the series expansion akin to the free-space Green function of Problem 2.17. Then show that it can be written in closed form as

$$G = \ln\left[\frac{\rho^2\rho'^2 + b^4 - 2\rho\rho' b^2\cos(\phi-\phi')}{b^2(\rho^2+\rho'^2-2\rho\rho'\cos(\phi-\phi'))}\right]$$

or

$$G = \ln\left[\frac{(b^2-\rho^2)(b^2-\rho'^2) + b^2|\rho-\rho'|^2}{b^2|\rho-\rho'|^2}\right].$$

(b) Show that the solution of the Laplace equation with the potential given as $\Phi(b,\phi)$ on the cylinder can be expressed as Poisson's integral of Problem 2.12.

(c) What changes are necessary for the Green function for the *exterior* problem ($b < \rho < \infty$), for both the Fourier expansion and the closed form? [Note that the exterior Green function is not rigorously correct because it does not vanish for ρ or $\rho' \to \infty$. For situations in which the potential falls of fast enough as $\rho \to \infty$, no mistake is made in its use.]

(a) As before, we write the general solution of the radial equation for g_m in the two distinct regions:

$$g_m(\rho,\rho') = \begin{cases} A_{1m}\rho'^m + B_{1m}\rho'^{-m} & , \quad \rho' < \rho \\ A_{2m}\rho'^m + B_{2m}\rho'^{-m} & , \quad \rho' > \rho. \end{cases} \tag{21}$$

The first boundary conditions are that g_m remain finite at the origin and vanish on the cylinder boundary. This requires that

$$B_{1m} = 0$$

and

$$A_{2m} b^m + B_{2m} b^{-m} = 0$$

so

$$A_{2m} = \gamma_m b^{-m} \qquad B_{2m} = -\gamma_m b^m$$

for some constant γ_m.

Next, g_m must be continuous at $\rho = \rho'$:

$$\begin{aligned} A_{1m}\rho^m &= \gamma_m \left[\left(\frac{\rho}{b}\right)^m - \left(\frac{b}{\rho}\right)^m \right] \\ A_{1m} &= \frac{\gamma_m}{\rho^m} \left[\left(\frac{\rho}{b}\right)^m - \left(\frac{b}{\rho}\right)^m \right]. \end{aligned}$$

With this we have

$$\begin{aligned} g_m(\rho,\rho') &= \gamma_m \left[\left(\frac{\rho}{b}\right)^m - \left(\frac{b}{\rho}\right)^m \right] \left(\frac{\rho'}{\rho}\right)^m \qquad , \qquad \rho' < \rho \\ &= \gamma_m \left[\left(\frac{\rho'}{b}\right)^m - \left(\frac{b}{\rho'}\right)^m \right] \qquad , \qquad \rho' > \rho. \end{aligned}$$

Finally, $dg_m/d\rho'$ must have a finite jump of magnitude $1/\rho$ at $\rho' = \rho$.

$$\begin{aligned} \frac{1}{\rho} &= \left.\frac{dg_m}{d\rho'}\right|_{\rho'=\rho_+} - \left.\frac{dg_m}{d\rho'}\right|_{\rho'=\rho_-} \\ &= m\gamma_m \left[\frac{\rho^{m-1}}{b^m} + \frac{b^m}{\rho^{m+1}} \right] - m\gamma_m \left[\left(\frac{\rho}{b}\right)^m - \left(\frac{b}{\rho}\right)^m \right] \frac{1}{\rho} \\ &= 2m\gamma_m \left(\frac{b}{\rho}\right)^m \frac{1}{\rho} \end{aligned}$$

so

$$\gamma_m = \frac{1}{2m}\left(\frac{\rho}{b}\right)^m$$

and

$$\begin{aligned} g_m(\rho,\rho') &= \frac{1}{2m} \left[\left(\frac{\rho\rho'}{b^2}\right)^m - \left(\frac{\rho'}{\rho}\right)^m \right] \qquad , \qquad \rho' < \rho \\ &= \frac{1}{2m} \left[\left(\frac{\rho\rho'}{b^2}\right)^m - \left(\frac{\rho}{\rho'}\right)^m \right] \qquad , \qquad \rho' > \rho. \end{aligned}$$

or

$$g_m(\rho,\rho') = \frac{1}{2m} \left[\left(\frac{\rho\rho'}{b^2}\right)^m - \left(\frac{\rho_<}{\rho_>}\right)^m \right].$$

Plugging into the expansion for G gives

$$G(\rho,\phi,\rho',\phi') = \frac{1}{2\pi} \sum_{n=1}^{\infty} \frac{1}{m} \left[\left(\frac{\rho\rho'}{b^2}\right)^m - \left(\frac{\rho_<}{\rho_>}\right)^m \right] \cos m(\phi - \phi'). \tag{22}$$

Here we need to work out an auxiliary result:

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{1}{n}x^n\cos n(\phi-\phi') &= \sum_{n=1}^{\infty}\left[\int_0^x u^{n-1}du\right]\cos m(\phi-\phi') \\
&= \int_0^x\left\{\frac{1}{u}\sum_{n=1}^{\infty}u^n\cos n(\phi-\phi')\right\}du \\
&= \int_0^x\left\{\frac{\cos(\phi-\phi')-u}{1+u^2-2u\cos(\phi-\phi')}\right\}du \\
&= -\frac{1}{2}\left|\ln(1-2u\cos(\phi-\phi')+u^2)\right|_0^x \\
&= -\frac{1}{2}\ln[1-2x\cos(\phi-\phi')+x^2].
\end{aligned}$$

(I summed the infinite series here back in Problem 2.12. The integral in the second-to-last step can be done by partial fraction decomposition, although I cheated and looked it up on www.integrals.com). We can apply this result individually to the two terms in (22):

$$\begin{aligned}
G(\rho,\phi;\rho',\phi') &= -\frac{1}{4\pi}\ln\left[\frac{1+(\rho\rho'/b^2)^2-2(\rho\rho'/b^2)\cos(\phi-\phi')}{1+(\rho_</\rho_>)^2-2(\rho_</\rho_>)\cos(\phi-\phi')}\right] \\
&= -\frac{1}{4\pi}\ln\left[\left(\frac{\rho_>^2}{b^4}\right)\frac{b^4+\rho^2\rho'^2-2\rho\rho'b^2\cos(\phi-\phi')}{\rho_>^2+\rho_<^2-2\rho_<\rho_>\cos(\phi-\phi')}\right] \\
&= -\frac{1}{4\pi}\ln\left[\left(\frac{\rho_>^2}{b^4}\right)\frac{b^4+\rho^2\rho'^2-2\rho\rho'b^2\cos(\phi-\phi')}{\rho_>^2+\rho_<^2-2\rho_<\rho_>\cos(\phi-\phi')}\right] \\
&= -\frac{1}{4\pi}\ln\left(\frac{\rho_>^2}{b^2}\right) \\
&\quad -\frac{1}{4\pi}\ln\left[\frac{b^4+\rho^2\rho'^2-2\rho\rho'b^2\cos(\phi-\phi')}{b^2(\rho^2+\rho'^2-2\rho\rho'\cos(\phi-\phi'))}\right]
\end{aligned} \tag{23}$$

This is Jackson's result, with an additional ln term thrown in for good measure. I'm not sure why Jackson didn't quote this term as part of his answer; he did include it in his answer to problem 2.17 (c). Did I do something wrong?

(b) Now we want to plug the expression for G above into (16) to compute the potential within the cylinder. If there is no charge inside the cylinder, the volume integral vanishes, and we are left with the surface integral:

$$\Phi(\rho,\phi) = \int\Phi(b,\phi')\left.\frac{\partial G}{\partial\rho'}\right|_{\rho'=b}dA'. \tag{24}$$

where the integral is over the surface of the cylinder.

For this we need the normal derivative of (23) on the cylinder:

$$\frac{\partial G}{\partial\rho'} = -\frac{1}{4\pi}\left\{\frac{2\rho^2\rho'-2\rho b^2\cos(\phi-\phi')}{b^4+\rho^2\rho'^2-2\rho\rho'b^2\cos(\phi-\phi')}-\frac{2\rho'-2\rho\cos(\phi-\phi')}{\rho^2+\rho'^2-2\rho\rho'\cos(\phi-\phi')}\right\}.$$

Evaluated at $\rho' = b$ this is

$$\left.\frac{\partial G}{\partial \rho'}\right|_{\rho'=b} = -\frac{1}{2\pi}\left\{\frac{\rho^2 - b^2}{b(\rho^2 + b^2 - 2\rho b\cos(\phi - \phi'))}\right\}.$$

In the surface integral, the extra factor of b on the bottom is cancelled by the factor of b in the area element dA', and (24) becomes just the result of Problem 2.12.

(c) For the exterior problem we again start with the solution (21). Now the boundary conditions are different; the condition at ∞ gives $A_{2m} = 0$, while the condition at b gives

$$A_{1m} = \gamma_m b^{-m} \qquad B_{1m} = -\gamma_m b^m.$$

From the continuity condition at $\rho' = \rho$ we find

$$A_{2m} = \gamma_m \rho^m \left[\left(\frac{\rho}{b}\right)^m - \left(\frac{b}{\rho}\right)^m\right].$$

The finite derivative jump condition gives

$$-m\gamma_m\left[\left(\frac{\rho}{b}\right)^m - \left(\frac{b}{\rho}\right)^m\right]\frac{1}{\rho} - m\gamma_m\left[\left(\frac{\rho}{b}\right)^m + \left(\frac{b}{\rho}\right)^m\right]\frac{1}{\rho} = \frac{1}{\rho}$$

or

$$\gamma_m = -\frac{1}{2m}\left(\frac{b}{\rho}\right)^m.$$

Putting it all together we have for the exterior problem

$$g_m = \frac{1}{2m}\left[\left(\frac{b^2}{\rho\rho'}\right)^m - \left(\frac{\rho_<}{\rho_>}\right)^m\right].$$

This is the same g_m we came up with before, but with b^2 and $\rho\rho'$ terms flipped in first term. But the closed-form expression was symmetrical in those two expressions (except for the mysterious ln term) so the closed-form expression for the exterior Green's function should be the same as the interior Green's function.

Solutions to Problems in Jackson, *Classical Electrodynamics*, Third Edition

Homer Reid

June 15, 2000

Chapter 3: Problems 1-10

Problem 3.1

Two concentric spheres have radii $a, b(b > a)$ and each is divided into two hemispheres by the same horizontal plane. The upper hemisphere of the inner sphere and the lower hemisphere of the outer sphere are maintained at potential V. The other hemispheres are at zero potential.
Detemine the potential in the region $a \le r \le b$ as a series in Legendre polynomials. Include terms at least up to $l = 4$. Check your solution against known results in the limiting cases $b \to \infty$ and $a \to 0$.

The expansion of the electrostatic potential in spherical coordinates for problems with azimuthal symmetry is

$$\Phi(r,\theta) = \sum_{l=0}^{\infty} \left[A_l r^l + B_l r^{-(l+1)} \right] P_l(\cos\theta). \qquad (1)$$

We find the coefficients A_l and B_l by applying the boundary conditions. Multiplying both sides by $P_{l'}(\cos\theta)$ and integrating from -1 to 1 gives

$$\int_{-1}^{1} \Phi(r,\theta) P_l(\cos\theta) d(\cos\theta) = \frac{2}{2l+1} \left[A_l r^l + B_l r^{-(l+1)} \right].$$

At $r = a$ this yields

$$V \int_0^1 P_l(x) dx = \frac{2}{2l+1} \left[A_l a^l + B_l a^{-(l+1)} \right],$$

and at $r = b$,

$$V \int_{-1}^{0} P_l(x)dx = \frac{2}{2l+1}\left[A_l b^l + B_l b^{-(l+1)}\right].$$

The integral from 0 to 1 vanishes for l even, and is given in the text for l odd:

$$\int_0^1 P_l(x)dx = (-\frac{1}{2})^{(l-1)/2}\frac{(l-2)!!}{2\left(\frac{l+1}{2}\right)!}.$$

The integral from -1 to 0 also vanishes for l even, and is just the above result inverted for l odd. This gives

$$\begin{aligned} V(-\frac{1}{2})^{(l-1)/2}\frac{(l-2)!!}{2\left(\frac{l+1}{2}\right)!} &= \frac{2}{2l+1}\left[A_l a^l + B_l a^{-(l+1)}\right] \\ -V(-\frac{1}{2})^{(l-1)/2}\frac{(l-2)!!}{2\left(\frac{l+1}{2}\right)!} &= \frac{2}{2l+1}\left[A_l b^l + B_l b^{-(l+1)}\right]. \end{aligned}$$

or

$$\begin{aligned} \alpha_l &= A_l a^l + B_l a^{-(l+1)} \\ -\alpha_l &= A_l b^l + B_l b^{-(l+1)} \end{aligned}$$

with

$$\alpha_l = V(-\frac{1}{2})a^{(l-1)/2}\frac{(2l+1)(l-2)!!}{4\left(\frac{l+1}{2}\right)!}.$$

The solution is

$$A_l = \alpha_l\left[\frac{b^{l+1}+a^{l+1}}{a^{2l+1}-b^{2l+1}}\right] \qquad B_l = -\alpha_l\left[\frac{a^{l+1}b^{l+1}(b^l+a^l)}{a^{2l+1}-b^{2l+1}}\right]$$

The first few terms of (1) are

$$\Phi(r,\theta) = \frac{3}{4}V\left[\frac{(a^2+b^2)r}{a^3-b^3} - \frac{a^2b^2(a+b)}{r^2(a^3-b^3)}\right]P_1(\cos\theta) - \frac{7}{16}\left[\frac{(a^4+b^4)r^3}{a^7-b^7} - \frac{a^4b^4(a^3+b^3)}{r^4(a^7-b^7)}\right]P_3(\cos\theta) + \cdots$$

In the limit as $b \to \infty$, the problem reduces to the exterior problem treated in Section 2.7 of the text. In that limit, the above expression becomes

$$\Phi(r,\theta) \to \frac{3}{4}V\left(\frac{a}{r}\right)^2 P_1(\cos\theta) - \frac{7}{16}V\left(\frac{a}{r}\right)^4 P_3(\cos\theta) + \cdots$$

in agreement with (2.27) with half the potential spacing. When $a \to 0$, the problem goes over to the *interior* version of the same problem, as treated in section 3.3 of the text. In that limit the above expression goes to

$$\Phi(r,\theta) \to -\frac{3}{4}V\left(\frac{r}{b}\right) P_1(\cos\theta) + \frac{7}{16}V\left(\frac{r}{b}\right)^3 P_3(\cos\theta) + \cdots$$

This agrees with equation (3.36) in the text, with the sign of V flipped, because here the more positive potential is on the lower hemisphere.

Problem 3.2

A spherical surface of radius R has charge uniformly distributed over its surface with a density $Q/4\pi R^2$, except for a spherical cap at the north pole, defined by the cone $\theta = \alpha$.

(a) Show that the potential inside the spherical surface can be expressed as

$$\Phi = \frac{Q}{8\pi\epsilon_0}\sum_{l=0}^{\infty}\frac{1}{2l+1}[P_{l+1}(\cos\alpha) - P_{l-1}(\cos\alpha)]\frac{r^l}{R^{l+1}}P_l(\cos\theta)$$

where, for $l = 0$, $P_{l-1}(\cos\alpha) = -1$. What is the potential outside?

(b) Find the magnitude and direction of the electric field at the origin.

(c) Discuss the limiting forms of the potential (part a) and electric field (part b) as the spherical cap becomes (1)very small, and (2) so large that the area with charge on it becomes a very small cap at the south pole.

(a) Let's denote the charge density on the sphere by $\sigma(\theta)$. At a point infinitesimally close to the surface of the sphere, the electric field is

$$\mathbf{F} = -\nabla\Phi = -\frac{\sigma}{\epsilon_0}\hat{\mathbf{r}}$$

so

$$\left.\frac{\partial\Phi}{\partial r}\right|_{r=R} = \frac{\sigma}{\epsilon_0}. \qquad (2)$$

The expression for the potential within the sphere must be finite at the origin, so the B_l in (1) are zero. Differentiating that expansion, (2) becomes

$$\frac{\partial}{\partial r}\Phi(r,\theta) = \sum_{l=1}^{\infty} lA_l r^{l-1}P_l(\cos\theta)$$

Multiplying by $P_{l'}$ and integrating at $r = R$ gives

$$\frac{1}{\epsilon_0}\int_{-1}^{1}\sigma(\theta)P_l(\cos\theta)d(\cos\theta) = \frac{2l}{2l+1}A_l R^{l-1}$$

so

$$A_l = \frac{2l+1}{2lR^{l-1}}\cdot\left(\frac{Q}{4\pi R^2\epsilon_0}\right)\int_{-1}^{\cos\alpha}P_l(x)dx.$$

To evaluate the integral we use the identity (eq. 3.28 in the text)

$$P_l(x) = \frac{1}{(2l+1)}\frac{d}{dx}[P_{l+1}(x) - P_{l-1}(x)]$$

so

$$\int_{-1}^{\cos\alpha} P_l(x)dx = \frac{1}{2l+1}\left[P_{l+1}(\cos\alpha) - P_{l-1}(\cos\alpha)\right].$$

(We used the fact that $P_{l+1}(-1) = P_{l-1}(-1)$ for all l.) With this we have

$$A_l = \frac{Q}{8\pi\epsilon_0 l R^{l+1}}\left[P_{l+1}(\cos\alpha) - P_{l-1}(\cos\alpha)\right]$$

so the potential expansion is

$$\Phi(r,\theta) = \frac{Q}{8\pi\epsilon_0}\sum_{l=1}^{\infty}\frac{1}{l}[P_{l+1}(\cos\alpha) - P_{l-1}(\cos\alpha)]\frac{r^l}{R^{l+1}}P_l(\cos\theta).$$

Within the body of the sum, I have an l where Jackson has a $2l+1$. Also, he includes the $l = 0$ term in the sum, corresponding to a constant term in the potential. I don't understand how he can determine that constant from the information contained in the problem; the information about the charge density only tells you the derivative of the potential. There's nothing in this problem that fixes the *value* of the potential on the surface beyond an arbitrary constant.

(b) The field at the origin comes from the $l = 1$ term in the potential:

$$\begin{aligned}
\mathbf{E}(\mathbf{r}=0) = -\nabla\Phi|_{\mathbf{r}=0} &= -\left|\frac{\partial\Phi}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial\Phi}{\partial\theta}\hat{\theta}\right|_{r=0} \\
&= -\frac{Q}{8\pi\epsilon_0 R^2}[P_2(\cos\alpha) - 1]\left[P_1(\cos\theta)\hat{\mathbf{r}} + \frac{d}{d\theta}P_1(\cos\theta)\hat{\theta}\right] \\
&= -\frac{Q}{8\pi\epsilon_0 R^2}\left[\frac{3}{2}\cos^2\alpha - \frac{3}{2}\right]\left[\cos\theta\hat{\mathbf{r}} - \sin\theta\hat{\theta}\right] \\
&= \frac{3Q\sin^2\alpha}{16\pi\epsilon_0 R^2}\hat{\mathbf{k}}.
\end{aligned}$$

The field points in the positive z direction. That makes sense, since a positive test charge at the origin would sooner fly up out through the uncharged cap than through any of the charged surface.

Problem 3.3

A thin, flat, conducting, circular disk of radius R is located in the $x - y$ plane with its center at the origin, and is maintained at a fixed potential V. With the information that the charge density on a disc at fixed potential is proportional to $(R^2 - \rho^2)^{-1/2}$, where ρ is the distance out from the center of the disc,

(a) show that for $r > R$ the potential is

$$\Phi(r,\theta,\phi) = \frac{2V}{\pi}\frac{R}{r}\sum_{l=0}^{\infty}\frac{(-1)^l}{2l+1}\left(\frac{R}{r}^{2l}\right)P_{2l}(\cos\theta)$$

(b) find the potential for $r < R$.

(c) What is the capacitance of the disk?

We are told that the surface charge density on the disk goes like

$$\begin{aligned}
\sigma(r) &= K(R^2 - r^2)^{-1/2} \\
&= \frac{K}{R}\left[1 + \frac{1}{2}\left(\frac{r}{R}\right)^2 + \frac{3\cdot 1}{(2!)(2\cdot 2)}\left(\frac{r}{R}\right)^4 + \frac{5\cdot 3\cdot 1}{(3!)(2\cdot 2\cdot 2)}\left(\frac{r}{R}\right)^6 + \cdots\right] \\
&= \frac{K}{R}\sum_{n=0}^{\infty}\frac{(2n-1)!!}{n!\cdot 2^n}\left(\frac{r}{R}\right)^{2n} \qquad (3)
\end{aligned}$$

for some constant K. From the way the problem is worded, I take it we're not supposed to try to figure out what K is explicitly, but rather to work the problem knowing only the form of (3).

At a point infinitesimally close to the surface of the disk (i.e., as $\theta \to \pi/2$), the component of $\nabla\Phi$ in the direction normal to the surface of the disk must be proportional to the surface charge. At the surface of the disk, the normal direction is the negative $\hat{\theta}$ direction. Hence

$$\frac{1}{r}\frac{\partial}{\partial\theta}\Phi(r,\theta)\bigg|_{\theta=(\pi/2)} = \pm\frac{\sigma}{\epsilon_0}. \qquad (4)$$

with the plus (minus) sign valid for Φ above (below) the disc.

For $r < R$ the potential expansion is

$$\Phi(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta). \qquad (5)$$

Combining (3), (4), and (5) we have

$$\sum_{l=0}^{\infty} A_l r^{l-1}\,\frac{d}{d\theta}P_l(\cos\theta)\bigg|_{\cos\theta=0} = \pm\frac{K}{R\epsilon_0}\sum_{n=0}^{\infty}\frac{(2n-1)!!}{n!\cdot 2^n}\left(\frac{r}{R}\right)^{2n}. \qquad (6)$$

For l even, dP_l/dx vanishes at $x = 0$. For l odd, I used some of the Legendre polynomial identities to derive the formula

$$\frac{d}{dx}P_{2l+1}(x)\bigg|_{x=0} = (-1)^l(2l+1)\frac{(2l-1)!!}{l! \cdot 2^l}.$$

This formula reminds one strongly of expansion (3). Plugging into (6) and equating coefficents of powers of r, we find

$$A_{2l+1} = \pm\frac{(-1)^l K}{(2l+1)R^{2l+1}\epsilon_0}$$

so

$$\Phi(r,\theta) = A_0 \pm \frac{K}{\epsilon_0}\sum_{l=1}^{\infty}\frac{(-1)^l}{2l+1}\left(\frac{r}{R}\right)^{2l+1}P_{2l+1}(\cos\theta).$$

I wrote A_0 explicitly because we haven't evaluated it yet–the derivative condition we used earlier gave no information about it. To find A_0, observe that, on the surface of the disk ($\cos\theta = 0$), all the terms in the above sum vanish (because $P_l(0)$ is 0 for odd l) so $\Phi = A_0$ on the disk. But $\Phi = V$ on the disk. Therefore, $A_0 = V$. We have

$$\Phi(r,\theta) = V \pm \frac{K}{\epsilon_0}\sum_{l=1}^{\infty}\frac{(-1)^l}{2l+1}\left(\frac{r}{R}\right)^{2l+1}P_{2l+1}(\cos\theta) \tag{7}$$

where the plus (minus) sign is good for θ less than (greater than)$\pi/2$. Note that the presence of that $\pm$ sign preserves symmetry under reflection through the z axis, a symmetry that is clearly present in the physical problem.

(a) For $r > R$, there is no charge. Thus the potential and its derivative must be continuous everywhere–we can't have anything like the derivative discontinuity that exists at $\theta = \pi/2$ for $r < R$. Since the physical problem is symmetric under a sign flip in $\cos\theta$, the potential expansion can only contain P_l terms for l even. The expansion is

$$\Phi(r,\theta) = \sum_{l=0}^{\infty} B_{2l}r^{-(2l+1)}P_{2l}(\cos\theta).$$

At $r = R$, this must match up with (7):

$$V \pm \frac{K}{\epsilon_0}\sum_{l=1}^{\infty}\frac{(-1)^l}{2l+1}P_{2l+1}(\cos\theta) = \sum_{l=0}^{\infty} B_{2l}R^{-(2l+1)}P_{2l}(\cos\theta).$$

Multiplying both sides by $P_{2l}(\cos\theta)\sin(\theta)$ and integrating gives

$$\begin{aligned} B_{2l}\frac{2R^{-(2l+1)}}{4l+1} &= V\int_{-1}^{1}P_l(x)dx + \frac{K}{\epsilon_0}\sum_{l=1}^{\infty}\frac{(-1)^l}{2l+1}\left\{-\int_{-1}^{0}P_{2l+1}(x)P_{2l}(x)dx + \int_0^1 P_{2l+1}(x)P_l(x)dx\right\} \\ &= 2V\delta_{l,0} + \frac{2K}{\epsilon_0}\sum_{l=1}^{\infty}\frac{(-1)^l}{2l+1}\int_0^1 P_{2l+1}(x)P_{2l}(x)dx. \end{aligned}$$

but I can't do this last integral.

Problem 3.4

> The surface of a hollow conducting sphere of inner radius a is divided into an *even number* of equal segments by a set of planes; their common line of intersection is the z axis and they are distributed uniformly in the angle ϕ. (The segments are like the skin on wedges of an apple, or the earth's surface between successive meridians of longitude.) The segments are kept at fixed potentials $\pm V$, alternately.
>
> **(a)** Set up a series representation for the potential inside the sphere for the general case of $2n$ segments, and carry the calculation of the coefficients in the series far enough to determine exactly which coefficients are different from zero. For the nonvanishing terms exhibit the coefficients as an integral over $\cos\theta$.
>
> **(b)** For the special case of $n = 1$ (two hemispheres) determine explicitly the potential up to and including all terms with $l = 3$. By a coordinate transformation verify that this reduces to result (3.36) of Section 3.3.

(a) The general potential expansion is

$$\Phi(r,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left[A_{lm}r^l + B_{lm}r^{-(l+1)}\right]Y_{lm}(\theta,\phi). \tag{8}$$

For the solution within the sphere, finiteness at the origin requires $B_{lm} = 0$. Multiplying by $Y^*_{l'm'}$ and integrating over the surface of the sphere we find

$$\begin{aligned} A_{lm} &= \frac{1}{a^l}\int \Phi(a,\theta,\phi)\,Y^*_{lm}(\theta,\phi)\,d\Omega \\ &= \frac{V}{a^l}\sum_{k=1}^{n}(-1)^k\int_0^{\pi}\int_{2(k-1)\pi/n}^{2k\pi/n} Y^*_{lm}(\theta,\phi)\sin\theta\,d\phi\,d\theta \\ &= \frac{V}{a^l}\left[\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}\right]^{1/2}\left\{\int_{-1}^{1}P_l^m(x)\,dx\right\}\sum_{k=1}^{n}(-1)^k\left\{\int_{2(k-1)\pi/n}^{2k\pi/n}e^{-im\phi}\,d\phi\right\}. \end{aligned} \tag{9}$$

The ϕ integral is easy:

$$\int_{2(k-1)\pi/n}^{2k\pi/n} e^{-im\phi}\,d\phi = -\frac{1}{im}\left[e^{-2imk\pi/n} - e^{-2im(k-1)\pi/n}\right].$$

This is to be summed from $k = 1$ to n with a factor of $(-1)^k$ thrown in:

$$\begin{aligned} \sum &= -\frac{1}{im}\left[(e^{-2m\pi i(1/n)}-1) - (e^{-2m\pi i(2/n)} - e^{-2m\pi i(1/n)}) + \cdots - (1 - e^{-2m\pi i((n-1)/n)})\right] \\ &= \frac{2}{im}\left\{1 - e^{-2m\pi i/n} + e^{2(-2m\pi i/n)} - e^{3(-2m\pi i/n)} + \cdots + e^{(n-1)(-2m\pi i/n)}\right\}. \end{aligned} \tag{10}$$

Putting $x = -\exp(-2m\pi i/n)$, the thing in braces is

$$1 + x + x^2 + x^3 + \cdots + x^{n-1} = \frac{1 - x^n}{1 - x} = \frac{1 - e^{-2m\pi i}}{1 + e^{-2im\pi/n}},$$

Note that the numerator vanishes. Thus the only way this thing can be nonzero is if the denominator also vanishes, which only happens if the exponent in the denominator equates to -1. This only happens if $m/n = 1/2, 3/2, 5/2, \cdots$. In that case, the $2m\pi i/n$ term in the exponent of the terms in (10) equates to πi, so all the terms with a plus sign in (10) come out to +1, while all the terms with a minus sign come out to -1, so all n terms add constructively, and (10) equates to

$$\sum = \begin{cases} \frac{2n}{im}, & m = n/2, 3n/2, 5n/2, \cdots \\ 0, & \text{otherwise.} \end{cases}$$

Then the expression (9) for the coefficients becomes

$$A_{lm} = \frac{2nV}{ima^l}\left[\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}\right]^{1/2}\int_{-1}^{1} P_l^m(x)dx, \quad m = \frac{n}{2}, \frac{3n}{2}, \cdots = 0, \quad \text{otherwise.}$$

(b) As shown above, the only terms that contribute are those with $m = n/2$, $m = 3n/2$, et cetera. Of course there is also the constraint that $m < l$. Then, with $n = 2$, up to $l = 3$ the only nonzero terms in the series (9) are those with $l = 1$, $m = \pm 1$, and $l = 3$, $m = \pm 1$ or ± 3. We need to evaluate the θ integral for these terms. We have

$$\begin{aligned}
\int_{-1}^{1} P_1^1(x)\,dx &= -\int_{-1}^{1}(1-x^2)^{1/2}\,dx = -\pi \\
\int_{-1}^{1} P_3^1(x)\,dx &= -\int_{-1}^{1}(1-x^2)^{1/2}\left[\frac{15}{2}x^2 - \frac{3}{2}\right]dx = -\frac{3\pi}{8} \\
\int_{-1}^{1} P_3^3(x)\,dx &= -15\int_{-1}^{1}(1-x^2)^{3/2}\,dx = -\frac{15\pi}{4}.
\end{aligned}$$

Using these results in (**??**), we have

$$\begin{aligned}
A_{1\pm1} &= \pm\frac{4\pi V i}{a}\left[\frac{3}{4\pi\cdot 2}\right]^{1/2} \\
A_{3\pm1} &= \pm\frac{3\pi V i}{2a^3}\left[\frac{7\cdot 2}{4\pi\cdot 4!}\right]^{1/2} \\
A_{3\pm3} &= \pm\frac{5\pi V i}{a^3}\left[\frac{7}{4\pi\cdot 6!}\right]^{1/2}
\end{aligned}$$

Now we can plug these coefficients into (8) to piece together the solution. This involves some arithmetic in combining all the numerical factors in each coefficient, which I have skipped here.

$$\Phi(r,\theta,\phi) = V\left[3\left(\frac{r}{a}\right)\sin\theta\sin\phi + \frac{7}{16}\left(\frac{r}{a}\right)^3\sin\theta(5\cos^2\theta-1)\sin\phi + \frac{7}{144}\left(\frac{r}{a}\right)^3\sin^3\theta\sin 3\phi + \cdots\right]$$

Problem 3.6

Two point charges q and $-q$ are located on the z azis at $z = +a$ and $z = -a$, respectively.

(a) Find the electrostatic potential as an expansion in spherical harmonics and powers of r for both $r > a$ and $r < a$.

(b) Keeping the product $qa = p/2$ constant, take the limit of $a \to 0$ and find the potential for $r \neq 0$. This is by definition a dipole along the z azis and its potential.

(c) Suppose now that the dipole of part b is surrounded by a *grounded* spherical shell of radius b concentric with the origin. By linear superposition find the potential everwhere inside the shell.

(a) First of all, for a point on the z axis the potential is

$$\begin{aligned}\Phi(z) &= \frac{q}{4\pi\epsilon_0}\left[\frac{1}{|z-a|} - \frac{1}{z+a}\right]\\ &= \frac{q}{4\pi\epsilon_0 z}\left[1 + \left(\frac{a}{z}\right) + \left(\frac{a}{z}\right)^2 + \cdots - \left(1 - \left(\frac{a}{z}\right) + \left(\frac{a}{z}\right)^2\cdots\right)\right]\\ &= \frac{q}{2\pi\epsilon_0 z}\left[\left(\frac{a}{z}\right) + \left(\frac{a}{z}\right)^3 + \cdots\right]\end{aligned}$$

for $z > a$. Comparing this with the general expansion $\Phi = \sum B_l r^{-(l+1)} P_l(\cos\theta)$ at $\theta = 0$ we can identify the B_ls and write

$$\Phi(r,\theta) = \frac{q}{2\pi\epsilon_0 r}\left[\left(\frac{a}{r}\right)P_1(\cos\theta) + \left(\frac{a}{r}\right)^3 P_3(\cos\theta) + \cdots\right]$$

for $r > a$. For $r < a$ we can just swap a and r in this equation.

(b)

$$\begin{aligned}\Phi(r,\theta) &= \frac{qa}{2\pi\epsilon_0 r^2}\left[P_1(\cos\theta) + \left(\frac{a}{r}\right)^2 P_3(\cos\theta) + \cdots\right]\\ &= \frac{p}{4\pi\epsilon_0 r^2}\left[P_1(\cos\theta) + \left(\frac{a}{r}\right)^2 P_3(\cos\theta) + \cdots\right]\\ &\to \frac{p}{4\pi\epsilon_0 r^2}\cos\theta \quad \text{as} \quad a \to 0.\end{aligned}$$

(c) When we put the grounded sphere around the two charges, a surface charge distribution forms on the sphere. Let's denote by Φ_s the potential due to this charge distribution alone (not including the potential of the dipole) and by Φ_d the potential due to the dipole. To calculate Φ_s, we pretend there are no charges within the sphere, in which case we have the general expansion (1), with $B_l = 0$ to keep us finite at the origin. The total potential is just the sum $\Phi_s + \Phi_d$:

$$\Phi(r,\theta) = \frac{p}{4\pi\epsilon_0 r^2}\cos\theta + \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta).$$

The condition that this vanish at $r = b$ ensures, by the orthogonality of the P_l, that only the $l = 1$ term in the sum contribute, and that

$$A_1 = -\frac{p}{4\pi\epsilon_0 b^3}.$$

The total potential inside the sphere is then

$$\Phi(r,\theta) = \frac{p}{4\pi\epsilon_0 b^2}\left(1 - \frac{r}{b}\right) P_1(\cos\theta).$$

Problem 3.7

Three point charges $(q, -2q, q)$ are located in a straight line with separation a and with the middle charge $(-2q)$ at the origin of a grounded conducting spherical shell of radius b, as indicated in the figure.

(a) Write down the potential of the three charges in the absence of the grounded sphere. Find the limiting form of the potential as $a \to 0$, but the product $qa^2 = Q$ remains finite. Write this latter answer in spherical coordinates.

(b) The presence of the grounded sphere of radius b alters the potential for $r < b$. The added potential can be viewed as caused by the surface-charge density induced on the inner surface at $r = b$ or by image charges located at $r > b$. Use linear superposition to satisfy the boundary conditions and find the potential everywhere inside the sphere for $r < a$ and $r > a$. Show that in the limit $a \to 0$,

$$\Phi(r,\theta,\phi) \to \frac{Q}{2\pi\epsilon_0 r^3}\left(1 - \frac{r^5}{b^5}\right) P_2(\cos\theta).$$

(a) On the z axis, the potential is

$$\begin{aligned}
\Phi(z) &= \frac{q}{4\pi\epsilon_0}\left[-\frac{2}{z} + \frac{1}{|z-a|} + \frac{1}{z+a}\right] \\
&= \frac{q}{4\pi\epsilon_0 r}\left[-2 + \left(1 + \left(\frac{a}{z}\right) + \left(\frac{a}{z}\right)^2 \cdots\right) + \left(1 - \left(\frac{a}{z}\right) + \left(\frac{a}{z}\right)^2 + \cdots\right)\right] \\
&= \frac{q}{2\pi\epsilon_0 z}\left[\left(\frac{a}{z}\right)^2 + \left(\frac{a}{z}\right)^4 + \cdots\right].
\end{aligned}$$

As before, from this result we can immediately infer the expression for the potential at all points:

$$\begin{aligned}\Phi(r,\theta) &= \frac{q}{2\pi\epsilon_0 r}\left[\left(\frac{a}{r}\right)^2 P_2(\cos\theta) + \left(\frac{a}{r}\right)^4 P_4(\cos\theta) + \cdots\right] \\ &= \frac{qa^2}{2\pi\epsilon_0 r^3}\left[P_2(\cos\theta) + \left(\frac{a}{r}\right)^2 P_4(\cos\theta) + \cdots\right] \\ &\to \frac{Q}{2\pi\epsilon_0 r^3} P_2(\cos\theta) \quad \text{as} \quad a \to 0 \end{aligned} \tag{11}$$

(b) As in the previous problem, the surface charges on the sphere produce an extra contribution Φ_s to the potential within the sphere. Again we can express Φ_s with the expansion (1) (with $B_l = 0$), and we add Φ_s to (11) to get the full potential within the sphere:

$$\Phi(r,\theta) = \frac{Q}{2\pi\epsilon_0 r^3} P_2(\cos\theta) + \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$$

From the condition that Φ vanish at $r = b$, we determine that only the $l = 2$ term in the sum contributes, and that

$$A_2 = -\frac{Q}{2\pi\epsilon_0 b^5}.$$

Then the potential within the sphere is

$$\Phi(r,\theta) = \frac{Q}{2\pi\epsilon_0 r^3}\left[1 - \left(\frac{r}{b}\right)^5\right] P_2(\cos\theta).$$

Problem 3.9

A hollow right circular cylinder of radius b has its axis coincident with the z axis and its ends at $z = 0$ and $z = L$. The potential on the end faces is zero, while the potential on the cylindrical surface is given as $V(\phi, z)$. Using the appropriate separation of variables in cylindrical coordinates, find a series solution for the potential anywhere inside the cylinder.

The general solution of the Laplace equation for problems in cylindrical coordinates consists of a sum of terms of the form

$$R(\rho)Q(\phi)Z(z).$$

The ϕ function is of the form

$$Q(\phi) = A\sin\nu\phi + B\cos\nu\phi$$

with ν an integer. The z function is of the form

$$Z(z) = Ce^{kz} + De^{-kz}.$$

In this case, Z must vanish at $z = 0$ and $z = L$, which means we have to take k imaginary, i.e.

$$Z(z) = C\sin(k_n z) \quad \text{with} \quad k_n = \frac{\pi n}{L} \quad , \quad n = 1, 2, 3, \cdots$$

With this form for Z, R must be taken to be of the form

$$R(\rho) = EI_\nu(k_n\rho) + FK_\nu(k_n\rho).$$

Since we're looking for the potential on the inside of the cylinder and there is no charge at the origin, the solution must be finite as $\rho \to 0$, which requires $F = 0$. Then the potential expansion becomes

$$\Phi(\rho, \phi, z) = \sum_{n=1}^{\infty}\sum_{\nu=0}^{\infty}\left[A_{n\nu}\sin\nu\phi + B_{n\nu}\cos\nu\phi\right]\sin(k_n z)I_\nu(k_n\rho). \tag{12}$$

Multiplying by $\sin\nu'\phi\sin k_{n'}z$ and integrating at $r = b$, we find

$$\int_0^L\int_0^{2\pi} V(\phi, z)\sin\nu\phi\sin(k_n z)\,d\phi\,dz = \frac{\pi L}{2}I_\nu(k_n b)A_{n\nu}$$

so

$$A_{n\nu} = \frac{2}{\pi L I_\nu(k_n b)}\int_0^L\int_0^{2\pi} V(\phi, z)\sin(\nu\phi)\sin(k_n z)\,d\phi\,dz. \tag{13}$$

Similarly,

$$B_{n\nu} = \frac{2}{\pi L I_\nu(k_n b)}\int_0^L\int_0^{2\pi} V(\phi, z)\cos(\nu\phi)\sin(k_n z)\,d\phi\,dz. \tag{14}$$

Problem 3.10

For the cylinder in Problem 3.9 the cylindrical surface is made of two equal half-cylinders, one at potential V and the other at potential $-V$, so that

$$V(\phi, z) = \begin{cases} V & \text{for} \quad -\pi/2 < \phi < \pi/2 \\ -V & \text{for} \quad \pi/2 < \phi < 3\pi/2 \end{cases}$$

(a) Find the potential inside the cylinder.

(b) Assuming $L >> b$, consider the potential at $z = L/2$ as a function of ρ and ϕ and compare it with two-dimensional Problem 2.13.

The potential expansion is (12) with coefficients given by (13) and (14). The relevant integrals are

$$\int_0^L\int_0^{2\pi} V(\phi, z)\sin(\nu\phi)\sin(k_n z)\,d\phi\,dz$$

$$\begin{aligned}
&= V\left\{\int_0^L \sin(k_n z)\,dz\right\}\left\{\int_{-\pi/2}^{\pi/2}\sin(\nu\phi)\,d\phi - \int_{\pi/2}^{3\pi/2}\sin(\nu\phi)\,d\phi\right\}\\
&= 0
\end{aligned}$$

$$\int_0^L\int_0^{2\pi} V(\phi,z)\cos(\nu\phi)\sin(k_n z)\,d\phi\,dz$$

$$\begin{aligned}
&= V\left\{\int_0^L \sin(k_n z)\,dz\right\}\left\{\int_{-\pi/2}^{\pi/2}\cos(\nu\phi)\,d\phi - \int_{\pi/2}^{3\pi/2}\cos(\nu\phi)\,d\phi\right\}\\
&= \frac{2V}{\nu k_n}\left\{|\sin\nu\phi|_{-\pi/2}^{\pi/2} - |\sin\nu\phi|_{\pi/2}^{3\pi/2}\right\} \quad \text{(n odd)}\\
&= \left\{\begin{array}{ccl} 0 & , & n \text{ or } \nu \text{ even}\\ 8V/k_n\nu & , & n \text{ odd, } \nu = 1,5,9,\cdots\\ -8V/k_n\nu & , & n \text{ odd, } \nu = 3,7,11,\cdots \end{array}\right.
\end{aligned}$$

Hence, from (13) and (14),

$$\begin{aligned}
A_{n\nu} &= 0\\
B_{n\nu} &= 0, & n \text{ or } \nu \text{ even}\\
&= (-1)^{(\nu-1)/2}\cdot 16V/(n\nu\pi^2 I_\nu(k_n b)), & n \text{ and } \nu \text{ odd}
\end{aligned}$$

The potential expansion is

$$\Phi(\rho,\theta,z) = \frac{16V}{\pi^2}\sum_{n,\nu}\frac{(-1)^{(\nu-1)/2}}{n\nu I_v(k_n b)}\cos(\nu\phi)\sin(k_n z)I_\nu(k_n\rho) \tag{15}$$

where the sum contains only terms with n and ν odd.

(b) At $z = L/2$ we have

$$\Phi(\rho,\theta,L/2) = \frac{16V}{\pi^2}\sum_{n,\nu}\frac{(-1)^{(n+\nu-2)/2}}{n\nu}\cos(\nu\phi)\frac{I_\nu(k_n\rho)}{I_\nu(k_n b)}.$$

As $L\to\infty$, the arguments to the I functions become small. Using the limiting form for I_ν quoted in the text as equation (3.102), we have

$$\Phi(\rho,\theta) = \frac{16V}{\pi^2}\sum_{n,\nu}\frac{(-1)^{(n+\nu-2)/2}}{n\nu}\cos(\nu\phi)\left(\frac{\rho}{b}\right)^\nu.$$

The sums over n and ν are now decoupled:

$$\begin{aligned}
\Phi(\rho,\theta) &= \frac{16V}{\pi^2}\left\{\sum_{n=0}^\infty\frac{(-1)^n}{2n+1}\right\}\left\{\sum_{\nu=0}^\infty\frac{(-1)^\nu}{2\nu+1}\cos(\nu\phi)\left(\frac{\rho}{b}\right)^\nu\right\}\\
&= \frac{16V}{\pi^2}\left\{\frac{\pi}{4}\right\}\left\{\sum_{\nu=0}^\infty\frac{(-1)^\nu}{2\nu+1}\cos(\nu\phi)\left(\frac{\rho}{b}\right)^\nu\right\}\\
&= \frac{4V}{\pi}\tan^{-1}\left(\frac{2\rho b\cos\phi}{b^2-\rho^2}\right)
\end{aligned}$$

This agrees with the result of Problem 2.13, with $V_1 = -V_2 = V$. The first series is just the Taylor series for $\tan^{-1}(x)$ at $x = 1$, so it sums to $\pi/4$. The second series can also be put into the form of the Taylor series for $\tan^{-1}(x)$, using tricks exactly analogous to what I did in my solution for Problem 2.13.

Solutions to Problems in Jackson, *Classical Electrodynamics*, Third Edition

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June 15, 2000

Chapter 3: Problems 11-18

Problem 3.11

A modified Bessel-Fourier series on the interval $0 \le \rho \le a$ for an arbitrary function $f(\rho)$ can be based on the "homogenous" boundary conditions:

$$\begin{aligned} &\text{At} \quad \rho = 0, \quad \rho J_\nu(k\rho)\frac{d}{d\rho}J_\nu(k'\rho) = 0 \\ &\text{At} \quad \rho = a, \quad \frac{d}{d\rho}\ln[J_\nu(k\rho)] = -\frac{\lambda}{a} \qquad (\lambda \text{ real}) \end{aligned}$$

The first condition restricts ν. The second condition yields eigenvalues $k = y_{\nu n}/a$, where $y_{\nu n}$ is the nth positive root of $x\,dJ_\nu(x)/dx + \lambda J_\nu(x) = 0$.

(a) Show that the Bessel functions of different eigenvalues are orthogonal in the usual way.

(b) Find the normalization integral and show that an arbitrary function $f(\rho)$ can be expanded on the interval in the modified Bessel-Fourier series

$$f(\rho) = \sum_{n=1}^{\infty} A_n J_\nu\left(\frac{y_{\nu n}}{a}\right)$$

with the coefficients A_n given by

$$A_n = \frac{2}{a^2}\left[\left(1 - \frac{\nu^2}{y_{\nu n}^2}\right)J_\nu^2(y_{\nu n}) + \left(\frac{dJ_\nu(y_{\nu n})}{dy_{\nu n}}\right)^2\right]^{-1}\int_0^a f(\rho)\rho J_\nu\left(\frac{y_{\nu n}\rho}{a}\right)d\rho.$$

(a) The function $J_\nu(k\rho)$ satisfies the equation

$$\frac{1}{\rho}\frac{d}{d\rho}\left[\rho\frac{d}{d\rho}J_\nu(k\rho)\right]+\left(k^2-\frac{\nu^2}{\rho^2}\right)J_\nu(k\rho)=0. \tag{1}$$

Multiplying both sides by $\rho J_\nu(k'\rho)$ and integrating from 0 to a gives

$$\int_0^a\left\{J_\nu(k'\rho)\frac{d}{d\rho}\left[\rho\frac{d}{d\rho}J_\nu(k\rho)\right]+\left(k^2\rho-\frac{\nu^2}{\rho}\right)J_\nu(k'\rho)J_\nu(k\rho)\right\}d\rho=0. \tag{2}$$

The first term on the left can be integrated by parts:

$$\int_0^a J_\nu(k'\rho)\frac{d}{d\rho}\left[\rho\frac{d}{d\rho}J_\nu(k\rho)\right]d\rho$$
$$=\left|\rho J_\nu(k'\rho)\frac{d}{d\rho}J_\nu(k\rho)\right|_0^a-\int_0^a\rho\left[\frac{d}{d\rho}J_\nu(k'\rho)\right]\left[\frac{d}{d\rho}J_\nu(k\rho)\right]d\rho. \tag{3}$$

One of the conditions we're given is that the thing in braces in the first term here vanishes at $\rho=0$. At $\rho=a$ we can invoke the other condition:

$$\begin{aligned}\left.\frac{d}{d\rho}\ln[J_\nu(k\rho)]\right|_{\rho=a} &= \left.\frac{1}{J_\nu(k\rho)}\frac{d}{d\rho}J_\nu(k\rho)\right|_{\rho=a}=-\frac{\lambda}{a}\\ &\to\ a\frac{d}{d\rho}J_\nu(ka)=-\lambda J_\nu(ka).\end{aligned}$$

Plugging this into (3), we have

$$\int_0^a J_\nu(k'\rho)\frac{d}{d\rho}\left[\rho\frac{d}{d\rho}J_\nu(k\rho)\right]d\rho$$
$$=-\lambda J_\nu(k'\rho)J_\nu(k\rho)-\int_0^a\rho\left[\frac{d}{d\rho}J_\nu(k'\rho)\right]\left[\frac{d}{d\rho}J_\nu(k\rho)\right]. \tag{4}$$

This is clearly symmetric in k and k', so when we write down (2) with k and k' switched and subtract from (2), the first integral (along with the ν^2/ρ term) vanishes, and we are left with

$$(k'^2-k^2)\int_0^a\rho J_\nu(k'\rho)J_\nu(k\rho)\,d\rho=0$$

proving orthogonality.

(b) If we multiply (1) by $\rho^2 J'(k\rho)$ and integrate, we find

$$\int_0^a\rho J_\nu'(k\rho)\frac{d}{d\rho}[\rho J_\nu'(k\rho)]d\rho+k^2\int_0^a\rho^2J_\nu(k\rho)J_\nu'(k\rho)d\rho-\nu^2\int_0^a J_\nu(k\rho)J_\nu'(k\rho)d\rho=0. \tag{5}$$

The first and third integrals are of the form $\int f(x)f'(x)dx$ and can be done immediately. In the second integral we put $f(\rho) = \rho^2 J_\nu(k\rho)$, $g'(\rho) = J'_\nu(k\rho)$ and integrate by parts:

$$\int_0^a \rho^2 J_\nu(k\rho)J'_\nu(k\rho)d\rho = \left|\rho^2 J_\nu^2(k\rho)\right|_0^a - 2\int_0^a \rho J_\nu^2(k\rho)d\rho - \int_0^a \rho^2 J_\nu(k\rho)J'_\nu(k\rho)d\rho$$

$$\rightarrow \int_0^a \rho^2 J_\nu(k\rho)J'_\nu(k\rho)d\rho = \frac{1}{2}a^2 J_\nu^2(ka) - \int_0^a \rho J_\nu^2(k\rho)d\rho.$$

Using this in (5),

$$\frac{a^2}{2}J_\nu'^2(ka) + \frac{(ak)^2}{2}aJ_\nu^2(ka) - k^2\int_0^a \rho J_\nu^2(k\rho)d\rho - \frac{\nu^2}{2}J_\nu^2(ka) = 0$$

so

$$\begin{aligned}\int_0^a \rho J_\nu^2(k\rho)d\rho &= \left(\frac{a^2}{2} - \frac{\nu^2}{2k^2}\right)J_\nu^2(ka) + \frac{a^2}{2k^2}J_\nu'^2(ka) \\ &= \frac{a^2}{2}\left\{\left(1 - \frac{\nu^2}{(ka)^2}\right)J_\nu^2(ka) + \left[\frac{d}{d(ka)}J_\nu(ka)\right]^2\right\}\end{aligned}$$

This agrees with what Jackson has if you note that k is chosen such that $ka = y_{nm}$.

Problem 3.12

An infinite, thin, plane sheet of conducting material has a circular hole of radius a cut in it. A thin, flat, disc of the same material and slightly smaller radius lies in the plane, filling the hole, but separated from the sheet by a very narrow insulating ring. The disc is maintained at a fixed potential V, whilc the infinite sheet is kept at zero potential.

(a) Using appropriate cylindrical coordinates, find an integral expression involving Bessel functions for the potential at any point above the plane.

(b) Show that the potential a perpendicular distance z above the *center* of the disc is

$$\Phi_0(z) = V\left(1 - \frac{z}{\sqrt{a^2+z^2}}\right)$$

(c) Show that the potential a perpendicular distance z above the *edge* of the disc is

$$\Phi_a(z) = \frac{V}{2}\left[1 - \frac{kz}{\pi a}K(k)\right]$$

where $k = 2a/(z^2+4a^2)^{1/2}$, and $K(k)$ is the complete elliptic integral of the first kind.

(a) As before, we can write the potential as a sum of terms $R(\rho)Q(\phi)Z(z)$. In this problem there is no ϕ dependence, so $Q = 1$. Also, the boundary conditions on Z are that it vanish at ∞ and be finite at 0, whence $Z(z) \propto \exp(-kz)$ for any k. Then the potential expansion becomes

$$\Phi(\rho, z) = \int_0^\infty A(k) e^{-kz} J_0(k\rho)\, dk. \tag{6}$$

To evaluate the coefficients $A(k)$, we multiply both sides by $\rho J_0(k'\rho)$ and integrate over ρ at $z = 0$:

$$\begin{aligned}\int_0^\infty \rho\Phi(\rho,0)J_0(k'\rho)\,d\rho &= \int_0^\infty A(k)\left\{\int_0^\infty \rho J_0(k\rho)J_0(k'\rho)\,d\rho\right\}dk \\ &= \frac{A(k')}{k'}\end{aligned}$$

so

$$\begin{aligned}A(k) &= k\int_0^\infty \rho\Phi(\rho,0)J_0(k\rho)\,d\rho \\ &= kV\int_0^a \rho J_0(k\rho)d\rho.\end{aligned}$$

Plugging this back into (6),

$$\Phi(\rho, z) = V\int_0^\infty \int_0^a k\rho' e^{-kz} J_0(k\rho)J_0(k\rho')\,d\rho'\,dk. \tag{7}$$

The ρ' integral can be done right away. To do it, I appealed to the differential equation for J_0:

$$J_0''(u) + \frac{1}{u}J_0'(u) + J_0(u) = 0$$

so

$$\begin{aligned}\int_0^x uJ_0(u)\,du &= -\int_0^x uJ_0''\,du - \int_0^x J_0'(u)\,du \\ &= -\left|uJ_0'(u)\right|_0^x + \int_0^x J_0'(u)\,du - \int_0^x J_0'(u)\,du \\ &= -\left|uJ_0'(u)\right|_0^x = -xJ_0'(x) = xJ_1(x).\end{aligned}$$

(In going from the first to second line, I integrated by parts.) Then (7) becomes

$$\Phi(\rho, z) = aV\int_0^\infty J_1(ka)J_0(k\rho)e^{-kz}\,dk. \tag{8}$$

(b) At $\rho = 0$, (7) becomes

$$\begin{aligned}\Phi(0,z) &= VJ_0(0)\int_0^a \rho' \left\{\int_0^\infty ke^{-kz}J_0(k\rho')dk\right\}d\rho' \\ &= V\int_0^a \rho' \left\{-\frac{\partial}{\partial z}\int_0^\infty e^{-kz}J_0(k\rho')dk\right\}d\rho' \\ &= V\int_0^a \rho' \left\{-\frac{\partial}{\partial z}\left(\frac{1}{\sqrt{\rho'^2+z^2}}\right)\right\}d\rho' \\ &= V\int_0^a \frac{z\rho'}{(\rho'^2+z^2)^{3/2}}d\rho'\end{aligned}$$

Here we substitute $u = \rho'^2 + z^2$, $du = 2\rho' d\rho$:

$$\begin{aligned}\Phi(0,z) &= \frac{VzJ_0(0)}{2}\int_{z^2}^{a^2+z^2} u^{-3/2}\,du \\ &= -Vz\left|\frac{1}{u^{1/2}}\right|_{z^2}^{a^2+z^2} \\ &= Vz\left[\frac{1}{z} - \frac{1}{\sqrt{z^2+z^2}}\right] \\ &= V\left[1 - \frac{z}{\sqrt{a^2+z^2}}\right]\end{aligned}$$

(b) At $\rho = a$, (8) becomes

$$\Phi(a,z) = aV\int_0^\infty J_1(ka)J_0(ka)e^{-kz}\,dk$$

Problem 3.13

Solve for the potential in Problem 3.1, using the appropriate Green function obtained in the text, and verify that the answer obtained in this way agrees with the direct solution from the differential equation.

For Dirichlet boundary value problems, the basic equation is

$$\Phi(\mathbf{x}) = -\frac{1}{\epsilon_0}\int_V G(\mathbf{x};\mathbf{x}')\rho(\mathbf{x}')\,dV' + \oint_S \Phi(\mathbf{x}')\left.\frac{\partial G(\mathbf{x};\mathbf{x}')}{\partial n}\right|_{x'} dA'. \tag{9}$$

Here there is no charge in the region of interest, so only the surface integral contributes. The Green's function for the two-sphere problem is

$$G(\mathbf{x};\mathbf{x}') = -\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{Y^*_{lm}(\theta',\phi')\,Y_{lm}(\theta,\phi)}{2l+1}R_l(r;r') \tag{10}$$

with

$$R_l(r;r') = \frac{1}{\left[1-\left(\frac{a}{b}\right)\right]^{2l+1}} \left(r_<^l - \frac{a^{2l+1}}{r_<^{l+1}} \right) \left(\frac{1}{r_>^{l+1}} - \frac{r_>^l}{b^{2l+1}} \right). \tag{11}$$

Actually in this case the potential cannot have any Φ dependence, so all terms with $m \neq 0$ in (10) vanish, and we have

$$G(\mathbf{x};\mathbf{x}') = -\frac{1}{4\pi} \sum_{l=0}^{\infty} P_l(\cos\theta) P_l(\cos\theta') R_l(r;r').$$

In this case the boundary surfaces are spherical, which means the normal to a surface element is always in the radial direction:

$$\frac{\partial}{\partial n} G(\mathbf{x};\mathbf{x}') = -\frac{1}{4\pi} \sum_{l=0}^{\infty} P_l(\cos\theta) P_l(\cos\theta') \frac{\partial}{\partial n} R_l(r;r').$$

The surface integral in (9) has two parts: one integral S_1 over the surface of the inner sphere, and a second integral S_2 over the surface of the outer sphere:

$$\begin{aligned}
S_1 &= -\frac{1}{4\pi} \sum_{l=0}^{\infty} P_l(\cos\theta) \left.\frac{\partial R_l}{\partial n}\right|_{r'=a} \left\{ \int_0^{\pi} \int_0^{2\pi} \Phi(a,\theta') P_l(\cos\theta') a^2 \sin\theta' \, d\phi \, d\theta' \right\} \\
&= -\frac{V}{2} \sum_{l=0}^{\infty} a^2 P_l(\cos\theta) \left.\frac{\partial R_l}{\partial n}\right|_{r'=a} \left\{ \int_0^1 P_l(x)\,dx \right\} \\
&= -\frac{V}{2} \sum_{l=0}^{\infty} a^2 \gamma_l P_l(\cos\theta) \cdot \left.\frac{\partial R_l}{\partial n}\right|_{r'=a}
\end{aligned}$$

where

$$\begin{aligned}
\gamma_l &= \int_0^1 P_l(x)\,dx \\
&= (-\frac{1}{2})^{(l-1)/2} \frac{(l-2)!!}{2[(l+1)/2]!}, \qquad l \text{ odd} \\
&= 0, \qquad l \text{ even.}
\end{aligned}$$

A similar calculation gives

$$\begin{aligned}
S_2 &= -\frac{V}{2} \sum_{l=0}^{\infty} b^2 P_l(\cos\theta) \left.\frac{\partial R_l}{\partial n}\right|_{r'=b} \left\{ \int_{-1}^{0} P_l(x)\,dx \right\} \\
&= \frac{V}{2} \sum_{l=0}^{\infty} b^2 \gamma_l P_l(\cos\theta) \left.\frac{\partial R_l}{\partial n}\right|_{r'=b}
\end{aligned}$$

because P_l is odd for l odd, so its integral from -1 to 0 is just the negative of the integral from 0 to 1. The final potential is the sum of S_1 and S_2:

$$\Phi(r,\theta) = \frac{V}{2} \sum_{l=0}^{\infty} \gamma_l P_l(\cos\theta) \left| r'^2 \frac{\partial R_l}{\partial n} \right|_{r'=a}^{r'=b} \tag{12}$$

Since the point of interest is always between the two spheres, to find the normal derivative at $r = a$ we differentiate with respect to $r_<$, and at $r = b$ with respect to $r_>$. Also, at $r = a$ the normal is in the $+r$ direction, while at $r = b$ the normal is in the negative r direction.

$$\begin{aligned} a^2 \frac{\partial}{\partial n} R_l(r;r')\bigg|_{r'=a} &= (2l+1)a^2 \frac{a^{l-1}}{\left[1-\left(\frac{a}{b}\right)\right]^{2l+1}} \left(\frac{1}{r^{l+1}} - \frac{r^l}{b^{2l+1}}\right) \\ b^2 \frac{\partial}{\partial n} R_l(r;r')\bigg|_{r'=b} &= (2l+1)b^2 \frac{b^{-(l+2)}}{\left[1-\left(\frac{a}{b}\right)\right]^{2l+1}} \left(r^l - \frac{a^{2l+1}}{r^{l+1}}\right) \end{aligned}$$

Combining these with some algebra gives

$$\Phi(r,\theta) = \frac{V}{2}\sum_{l=0}^{\infty}(2l+1)\gamma_l P_l(\cos\theta)\left[\frac{(ab)^{l+1}(b^l+a^l)r^{-(l+1)} - (a^{l+1}+b^{l+1})r^l}{b^{2l+1}-a^{2l+1}}\right]$$

in agreement with what we found in Problem 3.1.

Problem 3.14

> A line charge of length $2d$ with a total charge Q has a linear charge density varying as $(d^2 - z^2)$, where z is the distance from the midpoint. A grounded, conducting spherical shell of inner radius $b > d$ is centered at the midpoint of the line charge.
>
> **(a)** Find the potential everywhere inside the spherical shell as an expansion in Legendre polynomials.
>
> **(b)** Calculate the surface-charge density induced on the shell.
>
> **(c)** Discuss your answers to parts a and b in the limit that $d << b$.

First of all, we are told that the charge density $\rho(z) = \lambda(d^2 - z^2)$, and that the total charge is Q, whence

$$Q = 2\lambda \int_0^d (d^2 - z^2)dz = \frac{4}{3}d^3\lambda$$

$$\rightarrow \qquad \lambda = \frac{3Q}{4d^3}.$$

In this case we have azimuthal symmetry, so the Green's function is

$$G(\mathbf{x};\mathbf{x}') = -\frac{1}{4\pi}\sum_{l=0}^{\infty} P_l(\cos\theta')P_l(\cos\theta)R_l(r;r') \qquad (13)$$

with

$$R_l(r;r') = r_<^l \left(\frac{1}{r_>^{l+1}} - \frac{r_>^l}{b^{2l+1}} \right).$$

Since the potential vanishes on the boundary surface, the potential inside the sphere is given by

$$\Phi(r,\theta) = -\frac{1}{\epsilon_0}\int_V G(r,\theta;r',\theta')\rho(r',\theta')dV.$$

In this case ρ is only nonzero on the z axis, where $r = z$. Also, $P_l(\cos\theta)$=1 for $z > 0$, and $(-1)^l$ for $z < 0$. This means that the contributions to the integral from the portions of the line charge for $z > 0$ and $z < 0$ cancel out for odd l, and add constructively for even l:

$$\Phi(r,\theta) = \frac{1}{4\pi\epsilon_0}\sum_{l=0,2,4,\dots}^{\infty} P_l(\cos\theta)\left[2\int_0^d R_l(r;z)\rho(z)\,dz\right]$$

We have

$$\int_0^d R_l(r;z)\rho(z)\,dz = \lambda\int_0^d r_<^l \left(\frac{1}{r_>^{l+1}} - \frac{r_>^l}{b^{2l+1}} \right)(d^2-z^2)\,dz$$

This is best split up into two separate integrals:

$$= \lambda\int_0^d \frac{r_<^l}{r_>^{l+1}}(d^2-z^2)\,dz - \frac{\lambda}{b^{2l+1}}\int_0^d r_<^l r_>^l (d^2-z^2)\,dz$$

The second integral is symmetric between r and r', so we may integrate it directly:

$$\begin{aligned} -\frac{\lambda}{b^{2l+1}}\int_0^d r_<^l r_>^l (d^2-z^2)\,dz &= -\frac{\lambda r^l}{b^{2l+1}}\int_0^d z^l(d^2-z^2)\,dz \\ &= -\frac{\lambda r^l}{b^{2l+1}}\left[\frac{d^{l+3}}{l+1} - \frac{d^{l+3}}{l+3}\right] \\ &= -\frac{\lambda r^l d^{l+3}}{(l+1)(l+3)b^{2l+1}} \qquad (14) \end{aligned}$$

The first integral must be further split into two:

$$\lambda\int_0^d \frac{r_<^l}{r_>^{l+1}}(d^2-z^2)\,dz$$

$$
\begin{aligned}
&= \lambda \left\{ \frac{1}{r^{l+1}} \int_0^r z^l (d^2 - z^2)\, dz + r^l \int_r^d \frac{d^2 - z^2}{z^{l+1}}\, dz \right\} \\
&= \lambda \left\{ \frac{1}{r^{l+1}} \left[\frac{d^2 r^{l+1}}{l+1} - \frac{r^{l+3}}{l+3} \right] + r^l \left| - \frac{d^2}{l z^l} + \frac{1}{(l-2) z^{l-2}} \right|_r^d \right\} \\
&= \lambda \left\{ \frac{d^2}{l+1} - \frac{r^2}{l+3} + \left(\frac{r}{d} \right)^l d^2 \frac{2}{l(l+2)} - \frac{d^2}{l} + \frac{r^2}{l+2} \right\} \\
&= \lambda \left\{ \frac{r^2}{(l+2)(l+3)} - \frac{d^2}{l(l+1)} + \left(\frac{r}{d} \right)^l d^2 \frac{2}{l(l+2)} \right\}
\end{aligned}
$$

Combining this with (14), we have

$$
\int_0^d R_l(r;z)\rho(z)\, dz = \lambda \left\{ \frac{r^2}{(l+2)(l+3)} - \frac{d^2}{l(l+1)} + \left(\frac{r}{d} \right)^l d^2 \frac{2}{l(l+2)} - \frac{r^l d^{l+3}}{(l+1)(l+3) b^{2l+1}} \right\} \tag{15}
$$

But something is wrong here, because with this result the final potential will contain terms like $r^0 P_l(\cos\theta)$ and $r^2 P_l(\cos\theta)$, which do not satisfy the Laplace equation.

Problem 3.15

Consider the following "spherical cow" model of a battery connected to an external circuit. A sphere of radius a and conductivity σ is embedded in a uniform medium of conductivity σ'. Inside the sphere there is a uniform (chemical) force in the z direction acting on the charge carriers; its strength as an effective electric field entering Ohm's law is F. In the steady state, electric fields exist inside and outside the sphere and surface charge resides on its surface.

(a) Find the electric field (in addition to F) and current density everywhere in space. Determine the surface-charge density and show that the electric dipole moment of the sphere is $p = 4\pi\epsilon_0 \sigma a^3 F/(\sigma + 2\sigma')$.

(b) Show that the total current flowing out through the upper hemisphere of the sphere is

$$I = \frac{2\sigma\sigma'}{\sigma + 2\sigma'} \cdot \pi a^2 F$$

Calculate the total power dissipation outside the sphere. Using the lumped circuit relations, $P = I^2 R_e = I V_e$, find the effective external resistance R_e and voltage V_e.

(c) Find the power dissipated within the sphere and deduce the effective internal resistance R_i and voltage V_i.

(d) Define the total voltage through the relation $V_t = (R_e + R_i)I$ and show that $V_t = 4aF/3$, as well as $V_e + V_i = V_t$. Show that IV_t is the power supplied by the "chemical" force.

(a) What's going on in this problem is that the conductivity has a discontinuity going across the boundary of the sphere, but the current density must be constant there, which means there must an electric field discontinuity in inverse proportion to the conductivity discontinuity. To create this electric field discontinuity, there has to be some surface charge on the sphere, and this charge gives rise to extra fields both inside and outside the sphere.

Since there is no charge inside or outside the sphere, the potential in those two regions satisfied the Laplace equation, and may be expanded in Legendre polynomials:

$$\text{for } r < a, \qquad \Phi(r,\theta) = \Phi_{\text{in}}(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$$

$$\text{for } r > a, \qquad \Phi(r,\theta) = \Phi_{\text{out}}(r,\theta) = \sum_{l=0}^{\infty} B_l r^{-(l+1)} P_l(\cos\theta)$$

Continuity at $r = a$ requires that

$$A_l a^l = B_l a^{-l+1} \qquad \rightarrow \qquad B_l = a^{2l+1} A_l$$

so

$$\Phi(r,\theta) = \begin{cases} \Phi_{\text{in}}(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta), & r < a \\ \Phi_{\text{out}}(r,\theta) = \sum_{l=0}^{\infty} A_l a^{2l+1} r^{-(l+1)} P_l(\cos\theta), & r > a. \end{cases} \tag{16}$$

Now, in the steady state there can be no discontinuities in the current density, because if there were than there would be more current flowing into some region of space than out of it, which means charge would pile up in that region, which would be a growing source of electric field, which would mean we aren't in steady state. So the current density is continuous everywhere. In particular, the radial component of the current density is continuous across the boundary of the sphere, i.e.

$$J_r(r = a_-, \theta) = J_r(r = a_+, \theta). \tag{17}$$

Outside of the sphere, Ohm's law says that

$$\mathbf{J} = \sigma' \mathbf{E} = -\sigma' \nabla \Phi_{\text{out}}.$$

Inside the sphere, there is an extra term coming from the chemical force:

$$\mathbf{J} = \sigma(\mathbf{E} + F\hat{\mathbf{k}}) = \sigma(-\nabla\Phi_{\text{in}} + F\hat{\mathbf{k}}).$$

Applying (17) to these expressions, we have

$$\sigma\left(-\frac{\partial}{\partial r}\Phi_{\text{in}}\bigg|_{r=a} + F\cos\theta \right) = -\sigma' \, \frac{\partial}{\partial r}\Phi_{\text{out}}\bigg|_{r=a}$$

Using (16), this is

$$FP_1(\cos\theta) - \sum_{l=0}^{\infty} l A_l a^{l-1} P_l(\cos\theta) = \left(\frac{\sigma'}{\sigma}\right) \sum_{l=0}^{\infty} (l+1) A_l a^{l-1} P_l(\cos\theta).$$

Multiplying both sides by $P_{l'}(\cos\theta)$ and integrating from $-\pi$ to π, we find

$$F - A_1 = \left(\frac{\sigma'}{\sigma}\right) 2A_1 \tag{18}$$

for l=1, and

$$-lA_l = \left(\frac{\sigma'}{\sigma}\right)(l+1)A_l \tag{19}$$

$$\tag{20}$$

for $l \neq 1$. Since the conductivity ratio is positive, the second relation is impossible to satisfy unless $A_l = 0$ for $l \neq 1$. The first relation becomes

$$A_1 = \frac{\sigma}{\sigma + 2\sigma'}F.$$

Then the potential is

$$\Phi(r,\theta) = \begin{cases} \frac{\sigma}{\sigma+2\sigma'}Fr\cos\theta, & r < a \\ \frac{\sigma}{\sigma+2\sigma'}Fa^3r^{-2}\cos\theta, & r > a \end{cases} \tag{21}$$

The dipole moment $\mathbf{p}$ is defined by

$$\Phi(r,\theta) \to \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\mathbf{r}}{r^3} \qquad \text{as } r \to \infty. \tag{22}$$

The external portion of (21) can be written as

$$\Phi(r,\theta) = \frac{\sigma}{\sigma+2\sigma'}\frac{Fa^3z}{r^3}$$

and comparing this with (22) we can read off

$$\mathbf{p} = 4\pi\epsilon_0\frac{\sigma}{\sigma+2\sigma'}Fa^3\hat{\mathbf{k}}.$$

The electric field is found by taking the gradient of (21):

$$\mathbf{E}(r,\theta) = \begin{cases} -\frac{\sigma}{\sigma+2\sigma'}F\hat{\mathbf{k}}, & r < a \\ \frac{\sigma}{\sigma+2\sigma'}F\left(\frac{a}{r}\right)^3(2\cos\theta\hat{\mathbf{r}} + \sin\theta\hat{\theta}), & r > a \end{cases}$$

The surface charge $\sigma_s(\theta)$ on the sphere is proportional to the discontinuity in the electric field:

$$\begin{aligned} \sigma_s(\theta) &= \epsilon_0[E_r(r=a^+) - E_r(r=a^-)] \\ &= \frac{3\epsilon_0\sigma}{\sigma+2\sigma'}F\cos\theta. \end{aligned}$$

(b) The current flowing out of the upper hemisphere is just

$$\begin{aligned} \int \mathbf{J}\cdot\mathbf{dA} &= \sigma\int(\mathbf{E}_{\text{in}} + F\hat{\mathbf{k}})\cdot\mathbf{dA} \\ &= \sigma\left(1 - \frac{\sigma}{\sigma+2\sigma'}\right)F\int_0^{\pi/2}\int_0^{2\pi}\cos\theta\sin\theta\, a^2\, d\phi\, d\theta \\ &= 2\frac{\sigma\sigma'}{\sigma+2\sigma'}\cdot\pi a^2F \end{aligned} \tag{23}$$

The Ohmic power dissipation in a volume dV is

$$dP = \sigma E^2 dV \tag{24}$$

To see this, suppose we have a rectangular volume element with sides dx, dy, and dz. Consider first the current flowing in the x direction. The current density there is σE_x and the cross-sectional area is $dydz$, so $I = \sigma E_x dydz$. Also, the voltage drop in the direction of current flow is $V = E_x dx$. Hence the power dissipation due to current in the x direction is $IV = \sigma E_x^2 dV$. Adding in the contributions from the other two directions gives (24).

For the power dissipated outside the sphere we use the expression for the electric field we found earlier:

$$\begin{aligned}
P_{\text{out}} &= \sigma' \int_a^\infty \int_0^\pi \int_0^{2\pi} E^2(r,\theta,\phi) r^2 \sin\theta \, d\phi \, d\theta \, dr \\
&= 2\pi\sigma' \left(\frac{\sigma}{\sigma + 2\sigma'} \right)^2 F^2 a^6 \int_a^\infty \int_0^\pi \frac{1}{r^4} (4\cos^2\theta + \sin^2\theta) \sin\theta \, d\theta \, dr \\
&= \frac{8\pi}{3} \sigma' \left(\frac{\sigma}{\sigma + 2\sigma'} \right)^2 F^2 a^3
\end{aligned}$$

Dividing by (23), we find the effective external voltage V_e:

$$V_e = P_{\text{out}}/I = \frac{4}{3} aF \cdot \frac{\sigma}{\sigma + 2\sigma'}$$

and the effective external resistance:

$$R_e = P_{\text{out}}/I^2 = \frac{2}{3\pi a \sigma'}.$$

(c) The power dissipated inside the sphere is

$$\begin{aligned}
P_{\text{in}} = \sigma \int (\mathbf{E} + F\hat{\mathbf{k}})^2 dV &= \frac{4\sigma\sigma^{2'}}{(\sigma + 2\sigma')^2} F^2 \int dV \\
&= \frac{16\sigma\sigma^{2'}}{3(\sigma + 2\sigma')^2} \pi a^3 F^2
\end{aligned}$$

Since we're in steady state, the current flowing out through the upper hemisphere of the sphere must be replenished by an equal current flowing in through the lower half of the sphere, so to find the internal voltage and resistance we can just divide by (23):

$$\begin{aligned}
V_i &= P_{\text{in}}/I = \frac{8}{3} \frac{\sigma'}{\sigma + 2\sigma'} aF \\
R_i &= P_{\text{in}}/I^2 = \frac{4}{3\pi a \sigma}.
\end{aligned}$$

(c)

$$(R_e + R_i)I = \frac{2}{3\pi a}\left(\frac{1}{\sigma'} + \frac{2}{\sigma}\right) \cdot \frac{2\sigma\sigma'}{\sigma + 2\sigma'}\pi a^2 F = \frac{4}{3}aF$$

$$(V_i + V_e) = \frac{4aF}{3(\sigma + 2\sigma')}\sigma + 2\sigma' = \frac{4}{3}aF$$

Problem 3.17

The Dirichlet Green function for the unbounded space between the planes at $z = 0$ and $z = L$ allows discussion of a point charge or a distribution of charge between parallel conducting planes held at zero potential.

(a) Using cylindrical coordinates show that one form of the Green function is

$$G(\mathbf{x}, \mathbf{x}') = -\frac{1}{\pi L} \quad \times$$

$$\sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')} \sin\left(\frac{n\pi z}{L}\right) \sin\left(\frac{n\pi z'}{L}\right) I_m\left(\frac{n\pi \rho\prime_<}{L}\right) K_m\left(\frac{n\pi\rho_>}{L}\right).$$

(b) Show that an alternative form of the Green function is

$$G(\mathbf{x}, \mathbf{x}') = -\frac{1}{2\pi}\times$$

$$\sum_{m=-\infty}^{\infty} \int_0^{\infty} dk\, e^{im(\phi-\phi')} J_m(k\rho) J_m(k\rho') \frac{\sinh(kz_<)\sinh[k(L - z_>)]}{\sinh(kL)}.$$

In cylindrical coordinates, the solutions of the Laplace equation look like linear combinations of terms of the form

$$T_{mk}(\rho, \phi, z) = e^{im\phi} Z(kz) R_m(k\rho). \tag{25}$$

There are two possibilities for the combination $Z(kz)R_m(k\rho)$, both of which solve the Laplace equation:

$$Z(kz)R_m(k\rho) = (Ae^{kz} + Be^{-kz})[CJ_m(k\rho) + DN_m(k\rho)] \tag{26}$$

or

$$Z(kz)R_m(k\rho) = (Ae^{ikz} + Be^{-ikz})[CI_m(k\rho) + DK_m(k\rho)]. \tag{27}$$

The Green's function $G(\mathbf{x}; \mathbf{x}')$ must be a solution of the Laplace equation, and must thus take one of the above forms, at all points $\mathbf{x}' \neq \mathbf{x}$. At $\mathbf{x}' = \mathbf{x}$, G must be continuous, but have a finite discontinuity in its first derivative.

Furthermore, G must vanish on the boundary surfaces. These conditions may be met by dividing space into two regions, one on either side of the source point $\mathbf{x}$, and taking G to be different linear combinations of terms T (as in (25)) in the two regions. The question is, in which dimension (i.e., ρ, z, or ϕ) do we define the two "sides" of the source point?

(a) The first option is to imagine a cylindrical boundary at $\rho' = \rho$, i.e. at the radius of the source point, and take the inside and outside of the cylinder (i.e., $\rho' < \rho$ and $\rho' > \rho$) as the two distinct regions of space. Then, within each region, the entire range of z must be handled by one function, which means this one function must vanish at $z = 0$ and $z = L$. This cannot happen with terms of the form (26), so we are forced to take Z and R as in (27), with $B = -A$ and k restricted to the discrete values $k_n = n\pi/L$. Next considering the singularities of the ρ functions in (27), we see that, to keep G finite everywhere, for the inner region ($\rho' < \rho$) we can only keep the $I_m(k\rho)$ term, while for the outer region we can only keep the $K_m(k\rho)$ term. Then $G(\mathbf{x};\mathbf{x}')$ will consist of linear combinations of terms T as in (25) subject to the restrictions discussed above:

$$G(\mathbf{x};\mathbf{x}') = \begin{cases} \sum_{mn} A_{mn}(\mathbf{x}) e^{im\phi'} \sin(k_n z') I_m(k_n \rho'), & \rho' < \rho \\ \sum_{mn} B_{mn}(\mathbf{x}) e^{im\phi'} \sin(k_n z') K_m(k_n \rho'), & \rho' > \rho. \end{cases}$$

Clearly, to establish continuity at $\rho' = \rho$, we need to take $A_{mk}(\mathbf{x}) = \gamma_{mk}(z,\phi) K_m(k\rho)$ and $B_{mk}(\mathbf{x}) = \gamma_{mk}(z,\phi) I_m(k\rho)$, where γ_{mk} is any function of z and ϕ. Then we can write G as

$$G(\mathbf{x};\mathbf{x}') = \sum_{mk} \gamma_{mk}(z,\phi) e^{im\phi'} \sin(kz') I_m(k\rho_<) K_m(k\rho_>).$$

The obvious choice of γ_{mk} needed to make this a delta function in z and ϕ is $\gamma_{mk} = (4/L) e^{-im\phi} \sin(kz)$. Then we have

$$G(\mathbf{x};\mathbf{x}') = \frac{4}{L} \sum_{mk} e^{im(\phi'-\phi)} \sin(kz) \sin(kz') I_m(k\rho_<) K_m(k\rho_>).$$

What I don't quite understand is that this expression already has the correct delta function behavior in ρ, even though I never explicitly required this. To obtain this expression I first demanded that it satisfy the Laplace equation for all points $\mathbf{x}' \neq \mathbf{x}$, that it satisfy the boundary conditions of the geometry, and that it have the right delta function behavior in z' and ϕ'. But I never demanded that it have the correct delta function behavior in ρ', and yet it does. I guess the combination of the requirements that I did impose on this thing is already enough to ensure that it meets the final requirement.

(b) The second option is to imagine a plane boundary at $z' = z$, and take the two distinct regions to be the regions above and below the plane. In other words, the first region is that for which $0 \le z' \le z$, and the second region that for which $z \le z' \le L$. In this case, within each region the entire range of ρ' (from 0 to ∞) must be handled by one function. This requirement excludes terms of the form

(27), because K_m is singular at the origin, while I_m is singular at infinity, and there is no linear combination of these functions that will be finite over the whole range of ρ'. Hence we must use terms of the form (26). To ensure finiteness at the origin we must exlude the N_m term, so $D = 0$. To ensure vanishing at $z' = 0$ we must take $A = -B$, so the z' function in the region $0 \le z' \le z$ is proportional to $\sinh(kz')$. To ensure vanishing at $z' = L$ we must take $A = -Be^{-2kL}$, so the z' function in the region $z \le z' \le L$ is proportional to $\sinh[k(z' - L)]$. With these restrictions, the differential equation and the boundary conditions are satisfied for all terms of the form (25) with no limitation on k. Hence the Green's function will be an integral, not a sum, over these terms:

$$G(\mathbf{x}';\mathbf{x}) = \begin{cases} \sum_{m=0}^{\infty} \int_0^{\infty} A_m(k,\rho,\phi,z) e^{im\phi'} \sinh(kz') J_m(k\rho')\, dk, & 0 \le z' \le z \\ \sum_{m=0}^{\infty} \int_0^{\infty} B_m(k,\rho,\phi,z) e^{im\phi'} \sinh[k(z'-L)] J_m(k\rho')\, dk, & z \le z' \le L \end{cases}$$

Problem 3.18

The configuration of Problem 3.12 is modified by placing a conducting plane held at zero potential parallel to and a distance L away from the plane with the disc insert in it. For definiteness put the grounded plane at $z = 0$ and the other plane with the center of the disc on the z axis at $z = L$.

(a) Show that the potential between the planes can be written in cylindrical coordinates (z, ρ, ϕ) as

$$\Phi(z,\rho) = V \int_0^{\infty} d\lambda J_1(\lambda) J_0(\lambda\rho/a) \frac{\sinh(\lambda z/a)}{\sinh(\lambda L/a)}.$$

(b) Show that in the limit $a \to \infty$ with z, ρ, L fixed the solution of part a reduces to the expected result. Viewing your result as the lowest order answer in an expansion in powers of a^{-1}, consider the question of corrections to the lowest order expression if a is *large* compared to ρ and L, but not infinite. Are there difficulties? Can you obtain an explicit *estimate* of the corrections?

(c) Consider the limit of $L \to \infty$ with $(L - z)$, a and ρ fixed and show that the results of Problem 3.12 are recovered. What about corrections for $L \gg a$, but not $L \to \infty$?

(a) The general solution of the Laplace equation in cylindrical coordinates with angular symmetry that vanishes at $z = 0$ is

$$\Phi(\rho, z) = \int_0^{\infty} A(k) J_0(k\rho) \sinh(kz)\, dk. \qquad (28)$$

Multiplying both sides by $\rho J_0(k'\rho)$ and integrating at $z = L$ yields

$$\begin{aligned}\int_0^\infty \rho J_0(k'\rho)\Phi(\rho, L)\,d\rho &= \int_0^\infty A(k)\sinh(kL)\left\{\int_0^\infty \rho J_0(k'\rho)J_0(k\rho)\,d\rho\right\}\,dk\\ &= \int_0^\infty A(k)\sinh(kL)\left\{\frac{1}{k}\delta(k-k')\right\}\,dk\\ &= \frac{1}{k'}A(k')\sinh(k'L)\end{aligned}$$

so

$$\begin{aligned}A(k) &= \frac{k}{\sinh(kL)}\int_0^\infty \rho J_0(k\rho)\Phi(\rho, L)\,d\rho\\ &= \frac{Vk}{\sinh(kL)}\int_0^a \rho J_0(k\rho)\,d\rho\\ &= \frac{V}{k\sinh(kL)}\int_0^{ka} uJ_0(u)\,du. \qquad (29)\end{aligned}$$

I worked out this integral earlier, in Problem 3.12:

$$\int_0^x uJ_0(u)\,du = xJ_1(x).$$

Then (29) becomes

$$A(k) = \frac{V}{k\sinh(kL)}\cdot(ka)J_1(ka)$$

and (28) is

$$\begin{aligned}\Phi(\rho, z) &= V\int_0^\infty aJ_1(ka)J_0(k\rho)\frac{\sinh(kz)}{\sinh(kL)}\,dk\\ &= V\int_0^\infty J_1(\lambda)J_0(\lambda\rho/a)\frac{\sinh(\lambda z/a)}{\sinh(\lambda L/a)}\,d\lambda. \qquad (30)\end{aligned}$$

(b) For $x \ll 1$,

$$J_0(x) \to 1 - \frac{1}{4}x^2 + \cdots$$

and for $x \ll 1$ and $y \ll 1$,

$$\frac{\sinh(x)}{\sinh(y)} = \frac{x + \frac{1}{6}x^3 + \cdots}{y + \frac{1}{6}y^3 + \cdots} = \frac{x}{y}\left[1 + \frac{1}{6}(x^2 - y^2)\right] + O(x^4)$$

With these approximations we may expand the terms containing a in (30):

$$\begin{aligned}J_0(\lambda\rho/a)\frac{\sinh(\lambda z/a)}{\sinh(\lambda L/a)} &\approx \left(1 - \frac{1}{4}\left(\frac{\lambda\rho}{a}\right)^2\right)\left(\frac{z}{L}\right)\left(1 + \frac{1}{6}\left(\frac{\lambda}{a}\right)^2(x^2 - y^2)\right) \qquad (31)\\ &= \frac{z}{L}\left[1 - \left(\frac{\lambda}{a}\right)^2\left(\frac{1}{6}(L^2 - z^2) + \frac{1}{4}\rho^2\right) + \cdots\right] \qquad (32)\end{aligned}$$

Then the potential expansion (30) becomes

$$\Phi(\rho, z) = \frac{Vz}{L}\left[\int_0^\infty J_1(\lambda)\,d\lambda - \frac{1}{a^2}\left[\frac{1}{6}(L^2 - z^2) + \frac{1}{4}\rho^2\right]\int_0^\infty \lambda^2 J_1(\lambda)\,d\lambda + \cdots\right]$$

The first integral evaluates to 1, so for a infinite the potential becomes simply $\Phi(z) = Vz/L$. This is just what we expect to get for the potential between two infinite sheets, one grounded and the other at potential V.

The second integral, unfortunately, has a bit of an infinity problem. It's not hard to see where the problem comes: I derived the expansion above based on the premise that λ/a is small, but the integral goes over all λ up to ∞, so for any finite a the expansions eventually become invalid in the integral.

I'm still trying to work out a better procedure for estimating corrections for finite a.

(c) In this part we're interested in taking $L \to \infty$ and looking at the potential a fixed distance away from the plane with the circular insert. Calling the fixed distance z', the z coordinate of the point we're interested in is $L - z'$. We have

$$\begin{aligned}\frac{\sinh k(L - z')}{\sinh kL} &= \frac{\sinh(kL)\cosh(-kz') + \cosh(kL)\sinh(-kz')}{\sinh kL}\\ &= \cosh(kz') - \coth(kL)\sinh(kz') \end{aligned} \tag{33}$$

Now, $\coth(kL)$ differs significantly from 1 only for $kLa \lesssim 1$, in which region $kz' \lesssim z/L \ll 1$, so $\cosh(kz') \approx 1$ and $\sinh(kz') \approx 0$. By the time k gets big enough that kz' is starting to get significant, $\coth(kL)$ has long since started to look like 1, so the two terms in (33) add directly. The result is that, for all k, (33) can be approximated as $\exp(-kz')$. Then (30) becomes

$$\Phi(\rho, z) = aV\int_0^\infty J_1(ka)J_0(k\rho)e^{-kz'}\,dk$$

as we found in Problem 3.12.

Solutions to Problems in Jackson, *Classical Electrodynamics*, Third Edition

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August 6, 2000

Chapter 3: Problems 19-27

Problem 3.19

Consider a point charge q between two infinite parallel conducting planes held at zero potential. Let the planes be located at $z = 0$ and $z = L$ in a cylindrical coordinate system, with the charge on the z axis at $z = z_0$, $0 < z_0 < L$. Use Green's reciprocation theorem of Problem 1.12 with Problem 3.18 as the comparison problem.

(a) Show that the amount of induced charge on the plate at $z = L$ inside a circle of radius a whose center is on the z axis is given by

$$Q_L(a) = -\frac{q}{V}\Phi(z_0, 0)$$

(b) Show that the induced charge *density* on the upper plate can be written as

$$\sigma(\rho) = -\frac{q}{2\pi}\int_0^\infty dk\,\frac{\sinh(kz_0)}{\sinh(kL)}kJ_0(k\rho)$$

(c) Show that the charge density at $\rho = 0$ is

$$\sigma(0) = \frac{-\pi q}{8L^2}\sec^2\left(\frac{\pi z_0}{2L}\right)$$

(a) Green's reciprocation theorem says that

$$\int_V \rho'\Phi\,dV + \int_S \sigma'\Phi\,dA = \int_V \rho\Phi'\,dV + \int_S \sigma\Phi'\,dA. \tag{1}$$

We'll use the unprimed symbols to refer to the quantities of Problem 3.18, and the primed symbols to refer to those of Problem 3.19. Then

$$\begin{aligned}
\rho(r,z) &= 0 \\
\sigma(r,z) &= ? \\
\Phi(r,z) &= 0, && z = 0 \\
&= 0, && z = L \text{ and } r > a \\
&= V, && z = L \text{ and } r < a \\
&= V \int_0^\infty dk\, a J_1(ak) J_0(rk) \frac{\sinh(kz)}{\sinh(kL)} && 0 < z < L
\end{aligned}$$

$$\begin{aligned}
\rho'(r,z) &= q\delta(r)\delta(z - z_0) \\
\sigma'(r,z) &= ? \\
\Phi'(r,z) &= 0, && z = 0 \text{ or } z = L \\
&= ?, && 0 \le z \le L
\end{aligned}$$

Plugging into (1),

$$qV \int_0^\infty dk\, a J_1(ak) \frac{\sinh(kz_0)}{\sinh(kL)} + V \int_{z=L, r<a} \sigma'(r,z)\, dA = 0$$

so

$$\int_{z=L, r<a} \sigma'(r,z)\, dA = -q \int_0^\infty dk\, a J_1(ak) \frac{\sinh(kz_0)}{\sinh(kL)} = -\frac{q}{V}\Phi(z_0, 0) \qquad (2)$$

The integral on the left is just the total surface charge contained within a circle of radius a around the origin of the plane $z = L$.

(b) The integrand on the left of (2) doesn't depend on ϕ, so we can do the angular part of the integral right away to give

$$2\pi \int_0^a \sigma'(r,L) r\, dr = -q \int_0^\infty dk\, a J_1(ak) \frac{\sinh(kz_0)}{\sinh(kL)}$$

Differentiating both sides with respect to a, we have

$$2\pi a \sigma'(a,L) = -q \int_0^\infty dk\, \frac{\partial}{\partial a}[a J_1(ak)] \frac{\sinh(kz_0)}{\sinh(kL)} \qquad (3)$$

where I've blithely assumed that the partial derivative can be passed through the integral sign. The partial derivative is

$$\begin{aligned}
\frac{\partial}{\partial a}[a J_1(ak)] &= \left| \frac{\partial}{\partial x}[x J_1(x)] \right|_{x=ak} \\
&= |J_1(x) + x J_1'(x)|_{x=ak} \\
&= |x J_0(x)|_{x=ak} = ak J_0(ak)
\end{aligned}$$

so (3) becomes

$$\sigma'(a,L) = -\frac{q}{2\pi}\int_0^\infty dk\, kJ_0(ak)\frac{\sinh(kz_0)}{\sinh(kL)} \tag{4}$$

(c) At $a = 0$, (4) becomes

$$\sigma'(0,L) = \frac{-q}{2\pi}\int_0^\infty k\frac{\sinh(kz_0)}{\sinh(kL)}.$$

I have no idea how to do this integral.

Problem 3.22

The geometry of a two-dimensional potential problem is defined in polar coordinates by the surfaces $\phi = 0$, $\phi = \beta$, and $\rho = a$, as indicated in the sketch.
Using separation of variables in polar coordinates, show the the Green function can be written as

$$G(\rho,\phi;\rho',\phi') = \sum_{m=1}^{\infty} -\frac{1}{m\pi}\rho_<^{m\pi/\beta}\left(\frac{1}{\rho_>^{m\pi/\beta}} - \frac{\rho_>^{m\pi/\beta}}{a^{2m\pi/\beta}}\right)\sin\left(\frac{m\pi\phi}{\beta}\right)\sin\left(\frac{m\pi\phi'}{\beta}\right)$$

Problem 2.25 may be of use.

As before, the procedure for determining the Green's function is to split the region of interest into two parts (one on each 'side' of the observation point), find separate solutions of the Laplace equation that satisfy the boundary conditions in each region, and then join the two solutions at the source point such that their values match up but the first derivative (in whichever dimension we chose 'sides') has a finite discontinuity.

Suppose the observation point is (ρ,ϕ). Let's break the region into two subregions, defined by $0 \le \rho' \le \rho$ and $\rho \le \rho' \le a$. The general solution of the Laplace equation in two-dimensional polar coordinates is

$$\begin{aligned}\Phi(\rho',\phi') =& A_0 + B_0 \ln\rho' \\ &+ \sum_n \rho'^n[A_n \sin n\phi' + B_n\cos n\phi'] + \rho'^{-n}[C_n \sin n\phi' + D_n \cos n\phi'].\end{aligned}$$

The solution in the first region must be admissible down to $\rho' = 0$, which excludes the ln term and the negative powers of ρ. However, these terms may be included in the solution for the second region. In both regions, the solution must vanish at $\phi = 0$, which excludes the cos terms (i.e. $B_n = D_n = 0$). The solution must also vanish at $\phi = \beta$, which requires that $n = m\pi/\beta$, $m = 1, 2, \cdots$. With these considerations we may write down the solutions for G in the two regions:

$$G(\rho,\phi;\rho',\phi')$$

$$= \sum_{m=1}^{\infty} A_m \rho'^{m\pi/\beta} \sin\left(\frac{m\pi\phi'}{\beta}\right), \qquad 0 \le \rho' \le \rho \tag{5}$$

$$= \sum_{m=1}^{\infty} \left[B_m \rho'^{m\pi/\beta} + C_m \rho'^{-m\pi/\beta}\right] \sin\left(\frac{m\pi\phi'}{\beta}\right), \qquad \rho \le \rho' \le a \tag{6}$$

The solution in the second region must vanish at $\rho' = a$ for all ϕ', i.e.

$$B_m a^{m\pi/\beta} + C_m a^{-m\pi/\beta} = 0$$

so

$$B_m = \gamma_m a^{-m\pi/\beta} \quad \text{and} \quad C_m = -\gamma_m a^{m\pi/\beta}$$

where γ_m can be anything. Then (6) becomes

$$G(\rho,\phi;\rho',\phi') = \sum_{m=1}^{\infty} \gamma_m \left[\left(\frac{\rho'}{a}\right)^{m\pi/\beta} - \left(\frac{a}{\rho'}\right)^{m\pi/\beta}\right] \sin\left(\frac{m\pi\phi'}{\beta}\right) \qquad \rho \le \rho' \le a.$$

The solutions in the two regions must agree on the boundary between the two regions, i.e. at $\rho' = \rho$. This determines A_m and γ_m:

$$A_m = \lambda_m \left[\left(\frac{\rho}{a}\right)^{m\pi/\beta} - \left(\frac{a}{\rho}\right)^{m\pi/\beta}\right] \qquad \gamma_m = \lambda_m \rho^{m\pi/\beta}$$

where λ_m can be anything. Using these expressions for A_m, B_m, and C_m we can write

$$G(\rho,\phi;\rho',\phi')$$

$$= \sum_{m=1}^{\infty} \lambda_m \left[\left(\frac{\rho}{a}\right)^{m\pi/\beta} - \left(\frac{a}{\rho}\right)^{m\pi/\beta}\right] \rho'^{m\pi/\beta} \sin\left(\frac{m\pi\phi'}{\beta}\right) \qquad 0 \le \rho' \le \rho$$

$$= \sum_{m=1}^{\infty} \lambda_m \left[\left(\frac{\rho'}{a}\right)^{m\pi/\beta} - \left(\frac{a}{\rho'}\right)^{m\pi/\beta}\right] \rho^{m\pi/\beta} \sin\left(\frac{m\pi\phi'}{\beta}\right) \qquad \rho \le \rho' \le a.$$

This may be more succintly written as

$$G(\rho,\phi;\rho',\phi') = \sum_m \lambda_m f_m(\rho;\rho') \sin\left(\frac{m\pi\phi'}{\beta}\right) \tag{7}$$

where

$$f_m(\rho;\rho') = \left[\left(\frac{\rho_>}{a}\right)^{m\pi/\beta} - \left(\frac{a}{\rho_>}\right)^{m\pi/\beta}\right] \rho_<^{m\pi/\beta}.$$

The final step is to choose the constant λ_m in (7) such as to make

$$\nabla^2 G(\rho,\phi;\rho',\phi') = \frac{1}{\rho}\delta(\rho'-\rho)\delta(\phi'-\phi). \tag{8}$$

The Laplacian of (7) is

$$\nabla^2 G = \left[\frac{\partial^2}{\partial\rho'^2} + \frac{1}{\rho'^2}\frac{\partial^2}{\partial\phi'^2}\right] G = \sum_m \lambda_m \left[\frac{d^2}{d\rho'^2} f_m(\rho;\rho') - \left(\frac{m\pi}{\rho'\beta}\right)^2 f_m(\rho;\rho')\right] \sin\left(\frac{m\pi\phi'}{\beta}\right)$$

This is equal to (8) if

$$\lambda_m = \kappa_m \frac{1}{\beta}\sin\left(\frac{m\pi\phi}{\beta}\right) \tag{9}$$

and

$$\kappa_m \left[\frac{d^2}{d\rho'^2} f_m(\rho;\rho') - \left(\frac{m\pi}{\rho'\beta}\right)^2 f_m(\rho;\rho')\right] = \frac{1}{\rho}\delta(\rho'-\rho).$$

At all points $\rho' \neq \rho$, the latter condition is already satisfied by f as we constructed it earlier. At $\rho' = \rho$, the condition is achieved by choosing κ_m to satisfy

$$\kappa_m \left.\frac{d}{d\rho'} f_m(\rho;\rho')\right|_{\rho'=\rho-}^{\rho'=\rho+} = \frac{1}{\rho}. \tag{10}$$

Referring to (7), we have

$$\left.\frac{d}{d\rho'} f_m\right|_{\rho'+\rho+} = \frac{m\pi}{\beta}\left[\left(\frac{\rho}{a}\right)^{m\pi/\beta} + \left(\frac{a}{\rho}\right)^{m\pi/\beta}\right]\rho^{m\pi/\beta-1} \tag{11}$$

$$\left.\frac{d}{d\rho'} f_m\right|_{\rho'+\rho-} = \frac{m\pi}{\beta}\left[\left(\frac{\rho}{a}\right)^{m\pi/\beta} - \left(\frac{a}{\rho}\right)^{m\pi/\beta}\right]\rho^{m\pi/\beta-1}. \tag{12}$$

Subtracting (12) from (11) we obtain

$$\left.\frac{df_m}{d\rho'}\right|_{\rho'=\rho-}^{\rho'=\rho+} = \frac{2m\pi}{\beta}a^{m\pi/\beta}\cdot\frac{1}{\rho}.$$

Then from (10) we read off

$$\kappa_m = \frac{\beta}{2m\pi}a^{-m\pi/\beta}$$

and plugging this into (9) gives

$$\lambda_m = \frac{1}{2m\pi}a^{-m\pi/\beta}\sin\left(\frac{m\pi}{\beta}\right)\phi.$$

Plugging this into (7) we obtain finally

$$G(\rho,\phi;\rho',\phi') = \sum_m \frac{1}{2m\pi}\left[\left(\frac{\rho_<\rho_>}{a^2}\right)^{m\pi/\beta} - \left(\frac{\rho_<}{\rho_>}\right)^{m\pi/\beta}\right]\sin\left(\frac{m\pi\phi}{\beta}\right)\sin\left(\frac{m\pi\phi'}{\beta}\right)$$

I seem to be off by a factor of 2 here, but I can't find where.

Solutions to Problems in Jackson, *Classical Electrodynamics*, Third Edition

Homer Reid

September 5, 2000

Chapter 4: Problems 1-7

Problem 4.1

Calculate the multipole moments q_{lm} of the charge distributions shown as parts a and b. Try to obtain results for the nonvanishing moments valid for all l, but in each case find the first *two* sets of nonvanishing moments at the very least.

(c) For the charge distribution of the second set b write down the multipole expansion for the potential. Keeping only the lowest-order term in the expansion, plot the potential in the $x-y$ plane as a function of distance from the origin for distances greater than a.

(d) Calculate directly from Coulomb's law the exact potential for b in the $x-y$ plane. Plot it as a function of distance and compare with the result found in part c.

(a) The formula for the multipole moments is

$$q_{lm} = \sum_i q_i r_i^l Y_{lm}^*(\theta_i, \varphi_i)$$

where $(r_i, \theta_i, \varphi_i)$ is the location of the ith charge. Applying this to the first charge distribution, we have

$$\begin{aligned}
q_{lm} &= qa^l \left[Y_{lm}^*\left(\frac{\pi}{2}, 0\right) + Y_{lm}^*\left(\frac{\pi}{2}, \frac{\pi}{2}\right) - Y_{lm}^*\left(\frac{\pi}{2}, \pi\right) - Y_{lm}^*\left(\frac{\pi}{2}, \frac{3\pi}{2}\right) \right] \\
&= (-1)^l qa^l \left[\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!} \right]^{1/2} P_l^m(0) \left[1 + e^{-im\pi/2} - e^{-im\pi} - e^{-3im\pi/2} \right].
\end{aligned}$$

The last term in brackets vanishes for m even. For m odd, it evaluates to $(2-2i)$ for $m = -7, -3, 1, 5, 9, \cdots$, and to $(2+2i)$ for $m = -5, -1, 3, 7, \cdots$. Also, the $P_l^m(0)$ factor vanishes whenever l and m have different signs. These two things taken together mean that the only nonvanishing moments are those with l and m both odd. According to my calculations, the first few nonvanishing moments are

$$\begin{aligned} q_{1\pm1} &= qa\left(\frac{3}{4\pi}\right)^{1/2}(1\mp i) \\ q_{3\pm1} &= -qa^3\left(\frac{21}{\pi}\right)^{1/2}(1\mp i) \\ q_{3\pm3} &= qa^3\left(\frac{35}{16\pi}\right)^{1/2}(1\pm i). \end{aligned}$$

(b) In this case the charge at the origin only contributes to q_{00}. In all higher multipoles the r^l factor kills the contribution from this term. Hence we need only consider this charge when evaluating q_{00}, which clearly vanishes since the total charge is zero. Because the problem has azimuthal symmetry, all multipoles with $m \neq 0$ will vanish. With this in mind we may write down an expression for the nonvanishing multipole moments:

$$\begin{aligned} q_{l0} &= qa^l\left[Y_{l0}(0,0) + Y_{l0}(\pi,0)\right] \\ &= qa^l\left[\frac{2l+1}{4\pi}\right]^{1/2}\left[P_l(1) + P_l(-1)\right] \\ &= \begin{cases} 2qa^l\left[(2l+1)/4\pi\right]^{1/2}, & l \text{ even} \\ 0, & l \text{ odd.} \end{cases} \end{aligned} \tag{1}$$

The odd l terms vanish for this problem, which is as it must be, since the problem is clearly symmetrical about the xy plane.

(c) The potential expansion is

$$\Phi(r,\theta,\varphi) = \frac{1}{4\pi\epsilon_0 r}\sum_{lm}\frac{4\pi}{2l+1}q_{lm}\frac{Y_{lm}(\theta',\varphi')}{r^l}.$$

Plugging in expression (1) for q_{lm}, we have

$$\Phi(r,\theta) = \frac{q}{2\pi\epsilon_0 r}\sum_{l=1}P_{2l}(\cos\theta)\left(\frac{a}{r}\right)^{2l}$$

In particular, in the xy plane we have $\cos\theta = 0$, so

$$\Phi(r,\theta) = \frac{q}{2\pi\epsilon_0 r}\sum_{l=1}P_{2l}(0)\left(\frac{a}{r}\right)^{2l} = \sum_{l=1}(-1)^{2l+l}\frac{(2l+1)!!}{2^l l!}\left(\frac{a}{r}\right)^{2l}. \tag{2}$$

(d) On the other hand, working directly from Coulomb's law, we obtain for the potential in the xy plane

$$\begin{aligned}\Phi(\rho) &= \frac{1}{4\pi\epsilon_0}\left[-\frac{2q}{\rho} + \frac{2q}{(\rho^2+a^2)^{1/2}}\right] \\ &= \frac{q}{2\pi\epsilon_0\rho}\left[-1 + \left(1 + \left(\frac{a}{\rho}\right)^2\right)^{-1/2}\right] \\ &= \frac{q}{2\pi\epsilon_0\rho}\left[-1 + 1 - \frac{1}{2}\left(\frac{a}{\rho}\right)^2 + \frac{3}{8}\left(\frac{a}{\rho}\right)^4 - \cdots\right] \\ &= \frac{q}{2\pi\epsilon_0\rho}\sum_{l=1}(-1)^{2l+1}\frac{(2l+1)!!}{2^l l!}\left(\frac{a}{r}\right)^{2l}\end{aligned}$$

in agreement with equation (2).

Problem 4.2

A point dipole with dipole moment **p**is located at the point x_0. From the properties of the derivative of a Dirac delta function, show that for calculation of the potential Φ or the energy of a dipole in an external field, the dipole can be described by an effective charge density

$$\rho_{\text{eff}}(\mathbf{x}) = -\mathbf{p}\cdot\nabla\delta(\mathbf{x}-\mathbf{x}_0).$$

We have

$$\rho_{\text{eff}}(\mathbf{x}) = -\mathbf{p}\cdot\nabla\delta(\mathbf{x}-\mathbf{x}_0) = -p_x\frac{d}{dx}\delta(\mathbf{x}-\mathbf{x}_0) - p_y\frac{d}{dy}\delta(\mathbf{x}-\mathbf{x}_0) - p_z\frac{d}{dz}\delta(\mathbf{z}-\mathbf{z}_0).$$

The potential we would calculate from this charge density is

$$\begin{aligned}\Phi(\mathbf{x}) &= \frac{1}{4\pi\epsilon_0}\int_V \rho_{\text{eff}}(\mathbf{x}')\frac{1}{|\mathbf{x}-\mathbf{x}'|}dV' \\ &= \frac{1}{4\pi\epsilon_0}\left\{-p_x\int_V\left[\frac{d}{dx}\delta(\mathbf{x}'-\mathbf{x}_0)\right]\frac{1}{|\mathbf{x}-\mathbf{x}'|}dV' + \cdots\right\}\end{aligned} \qquad (3)$$

The integral can be evaluated by parts:

$$\begin{aligned}
&\int \left[\frac{d}{dx}\delta(\mathbf{x}'-\mathbf{x}_0)\right]\frac{1}{|\mathbf{x}-\mathbf{x}'|}dx' \\
&= \delta(\mathbf{x}'-\mathbf{x}_0)\frac{1}{|\mathbf{x}-\mathbf{x}'|} - \int \delta(\mathbf{x}'-\mathbf{x}_0)\frac{d}{dx}\frac{1}{|\mathbf{x}-\mathbf{x}'|}dx' \\
&= -\frac{d}{dx}\left.\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right|_{\mathbf{x}'=\mathbf{x}_0} \\
&= -\frac{(x-x_0)}{|\mathbf{x}-\mathbf{x}_0|^3}\hat{\mathbf{i}}.
\end{aligned}$$

(The surface term in the second line vanishes because of the δ function). Putting this back in (3) gives

$$\begin{aligned}
\Phi(\mathbf{x}) &= \frac{1}{4\pi\epsilon_0}\left\{\frac{p_x(x-x_0)}{|\mathbf{x}-\mathbf{x}_0|^3} + \cdots\right\} \\
&= \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}_0)}{|\mathbf{x}-\mathbf{x}_0|^3}
\end{aligned}$$

which is the standard expression for the potential of a dipole $\mathbf{p}$ at $\mathbf{x}_0$.

Problem 4.5

A localized charge density $\rho(x,y,z)$ is placed in an external electrostatic field described by a potential $\Phi^{(0)}(x,y,z)$. The external potential varies slowly in space over the region where the charge density is different from zero.

(a) From first principles calculate the total *force* acting on the charge distribution as an expansion in multipole moments times derivatives of the electric field, up to and including the quadrupole moments. Show that the force is

$$\mathbf{F} = q\mathbf{E}^0(0) + \left\{\nabla\left[\mathbf{p}\cdot\mathbf{E^0}(\mathbf{x})\right]\right\}_{x=0} + \left\{\nabla\left[\frac{1}{6}\sum_{j,k} Q_{jk}\frac{\partial E_j^0}{\partial x_k}(\mathbf{x})\right]\right\}_0 + \cdots$$

(b) Repeat the calculation of part a for the total *torque*. For simplicity, evaluate only one Cartesian component of the torque, say N_1. Show that this component is

$$N_1 = \left[\mathbf{p}\times\mathbf{E}^0(0)\right]_1 + \frac{1}{3}\left[\frac{\partial}{\partial x_3}\left(\sum_j Q_{2j}E_j^0\right) - \frac{\partial}{\partial x_2}\left(\sum_j Q_{3j}E_j^0\right)\right]_0 + \cdots$$

(a) The total force on the charge distribution coming from the external field is

$$\mathbf{F} = \sum_i q_i \mathbf{E}^0(\mathbf{r}_i)$$

where the sum is over all charges in the distribution, and $\mathbf{r}_i$ is the position of the ith charge. Looking at just the x component of this, we have

$$\begin{aligned}
F_1 &= \sum_i q_i E_1^0(\mathbf{r}_i) \\
&= \sum_i q_i \left\{ E_1^0(0) + \mathbf{r}_i \cdot \nabla E_1^0(x)\Big|_{\mathbf{x}=0} + \frac{1}{2}\sum_{jk} x_{ij}x_{ik}\frac{\partial^2 E_1^0}{\partial x_j \partial x_k} + \cdots \right\} \\
&= E_1(0)\cdot \sum_i q_i + \nabla E_1(x)|_{\mathbf{x}=0} \cdot \sum_i q_i \mathbf{r}_i + \sum_{jk} \frac{\partial^2 E_1}{\partial x_j \partial x_k} \cdot \left\{ \sum_i q_i x_{ij} x_{ik} \right\} + \cdots
\end{aligned} \tag{4}$$

(In the last line I dropped the 0 superscript on E; we'll just understand that E refers to the external field).

The sum in the first term of (4) is just the total charge q. The second term is

$$\begin{aligned}
\nabla E_1(x)|_{\mathbf{x}=0} \cdot \sum_i q_i \mathbf{r}_i &= \nabla E_1(x)|_{\mathbf{x}=0} \cdot \mathbf{p} \\
&= p_1 \frac{\partial E_1}{\partial x_1} + p_2 \frac{\partial E_1}{\partial x_2} + p_3 \frac{\partial E_1}{\partial x_3}
\end{aligned} \tag{5}$$

We now wish to invoke the fact that $\nabla \times \mathbf{E} = 0$ for an electrostatic field, so that

$$\frac{\partial E_1}{\partial x_2} = \frac{\partial E_2}{\partial x_1} \quad \text{and} \quad \frac{\partial E_1}{\partial x_3} = \frac{\partial E_3}{\partial x_1}. \tag{6}$$

Then (5) may be rewritten:

$$\nabla E_1(x)| \cdot \mathbf{p} = p_1 \frac{\partial E_1}{\partial x_1} + p_2 \frac{\partial E_2}{\partial x_1} + p_3 \frac{\partial E_3}{\partial x_1} = \frac{\partial}{\partial x}\left[\mathbf{p}\cdot E_1\right]_{\mathbf{x}=0}.$$

Now for the third term in (4). We can write this term out explicitly:

$$\frac{1}{2}\sum_i q_i \left\{ x_{i1}^2 \frac{\partial^2 E_1}{\partial^2 x_1} + x_{i2}^2 \frac{\partial^2 E_1}{\partial^2 x_2} + x_{i3}^2 \frac{\partial^2 E_1}{\partial^2 x_3} + 2x_{i1}x_{i2}\frac{\partial^2 E_1}{\partial x_1 \partial x_2} + 2x_{i1}x_{i3}\frac{\partial^2 E_1}{\partial x_1 \partial x_3} + 2x_{i1}x_{i3}\frac{\partial^2 E_1}{\partial x_2 \partial x_3} \right\}$$

where all the derivatives are understood to be evaluated at $\mathbf{x} = 0$. Next, using the relations (6), we can make sure that each term has at least one derivative with respect to x_1, and we can then pull the x_1 derivative operator outside the brackets:

$$= \frac{1}{2}\sum_i q_i \frac{\partial}{\partial x_1} \left\{ x_{i1}^2 \frac{\partial E_1}{\partial x_1} + x_{i2}^2 \frac{\partial E_2}{\partial x_2} + x_{i3}^2 \frac{\partial E_3}{\partial x_3} + 2x_{i1}x_{i2}\frac{\partial E_1}{\partial x_2} + 2x_{i1}x_{i3}\frac{\partial E_1}{\partial x_3} + 2x_{i2}x_{i3}\frac{\partial E_2}{\partial x_3} \right\}\Bigg|_{\mathbf{x}=0}.$$

Finally, we may add within the brackets the term $-r_i^2\nabla\cdot\mathbf{E}$, since this is zero anyway. This leaves us with

$$\begin{aligned}
&\frac{1}{2}\sum_i q_i \frac{\partial}{\partial x_1}\left\{(x_{i1}^2-\frac{1}{3}r_i^2)\frac{\partial E_1}{\partial x_1}+(x_{i2}^2-\frac{1}{3}r_i^2)\frac{\partial E_2}{\partial x_2}++(x_{i3}^2-\frac{1}{3}r_i^2)\frac{\partial E_3}{\partial x_3}\right.\\
+&\qquad \left. 2x_{i1}x_{i2}\frac{\partial E_1}{\partial x_2}+2x_{i1}x_{i3}\frac{\partial E_1}{\partial x_3}+2x_{i2}x_{i3}\frac{\partial E_2}{\partial x_3}\right\}\\
=&\quad \frac{1}{6}\frac{\partial}{\partial x_1}\sum_{jk}Q_{jk}\frac{\partial E_j}{\partial x_k}
\end{aligned}$$

which is the advertised result.

Problem 4.6

A nucleus with quadrupole moment Q finds itself in a cylindically symmetric electric field with a gradient $(\partial E_z/\partial z)_0$ along the z axis at the position of the nucleus.

(a) Show that the energy of quadrupole interaction is

$$W=-\frac{e}{4}Q\left(\frac{\partial E_z}{\partial z}\right)_0$$

(b) If it is known that $Q = 2\times 10^{-28}$ m^2 and that W/h is 10 MHz, where h is Planck's constant, calculate $(\partial E_z/\partial z)_0$ in units of $e/4\pi\epsilon_0 a_0^3$, where $a_0 = 4\pi\epsilon_0\hbar^2/me^2 = 0.529\times 10^{-10}$ m is the Bohr radius in hydrogen.

(c) Nuclear charge distributions can be approximated by a constant charge density throughout a spheroidal volume of semimajor axis a and semiminor axis b. Calculate the quadrupole moment of such a nucleus, assuming that the total charge is Ze. Given that Eu^{153} (Z=63) has a quadrupole moment $Q = 2.5\times 10^{-28}$ m^2 and a mean radius

$$R=(a+b)/2=7\times 10^{-15}\text{ m},$$

determine the fractional difference in radius $(a-b)/R$.

(a) To say that the nucleus has quadrupole moment Q means that the components of its Q_{ij} tensor are

$$Q_{11}=Q_{22}=-\frac{e}{2}Q,\qquad Q_{33}=eQ \tag{7}$$

with all other components vanishing. The energy of the dipole's interaction with the field gradient is given by Jackson's equation (4.24):

$$W=-\frac{1}{6}\sum_{ij}Q_{ij}\left|\frac{\partial E_j}{\partial x_i}\right|_{\mathbf{x}=0}$$

Plugging (7) into this, we have

$$W = -\frac{eQ}{6}\left[-\frac{1}{2}\frac{\partial E_1}{\partial x_1} - \frac{1}{2}\frac{\partial E_2}{\partial x_2} + \frac{\partial E_3}{\partial x_3}\right] \tag{8}$$

but since $\nabla \cdot \mathbf{E} = 0$, we have

$$\frac{\partial E_1}{\partial x_1} + \frac{\partial E_2}{\partial x_2} = -\frac{\partial E_3}{\partial x_3}$$

and inserting this into (8) gives

$$W = -\frac{eQ}{6}\left[\frac{3}{2}\frac{\partial E_3}{\partial x_3}\right] = -\frac{eQ}{4}\left|\frac{\partial E_3}{\partial x_3}\right|_{\mathbf{x}=0}.$$

(b)

$$\begin{aligned}\left(\frac{\partial E_z}{\partial z}\right) &= -\frac{4W}{Qe} = -\frac{4 \cdot 10 \text{ MHz } \cdot h}{(2 \times 10^{-28} \text{ m}^2)e} \\ &= -\frac{4 \cdot 10^7 \text{ s}^{-1} \cdot 2\pi e}{(2 \times 10^{-28} \text{ m}^2)(4\pi\epsilon_0 \alpha c)}\end{aligned}$$

(Here I have used the fine structure constant $\alpha = e^2/4\pi\epsilon_0\hbar c$.)

$$\begin{aligned}&= -\frac{4 \cdot 10^7 \text{ s}^{-1} \cdot 2\pi e \cdot 137}{(4\pi\epsilon_0)(2 \times 10^{-28} \text{ m}^2)(3 \times 10^8 \text{ m} \cdot \text{s}^{-1})} \\ &= \frac{e}{(4\pi\epsilon_0)(1.74 \times 10^{-30} \text{ m}^3)} \times \frac{0.148 \times 10^{-30} \text{ m}^3}{a_0^3} \\ &= 0.085 \times \frac{e}{4\pi\epsilon_0 a_0^3}.\end{aligned}$$

(c) An oblate spheroid with semimajor axis a and semiminor axis b has volume $V = 4\pi ab^2/3$. If we assume a charge Ze is distributed evenly throughout this volume, the charge density is

$$\rho = \frac{3Ze}{4\pi ab^2}$$

Taking the z axis along the major axis of the spheroid, the equation for the surface of the spheroid in cylindrical coordinates is

$$\left(\frac{z}{a}\right)^2 + \left(\frac{\rho}{b}\right)^2 = 1.$$

Then the 33 component of the quadrupole moment tensor is given by

$$\begin{aligned}
Q_{33} &= \int_{-a}^{a}\int_{0}^{[b^2-\frac{b^2z^2}{a^2}]^{1/2}}\int_0^{2\pi}\rho(3z^2-r^2)r\,d\theta\,dr\,dz \\
&= 2\pi\rho\int_{-a}^{a}\int_{0}^{[b^2-\frac{b^2z^2}{a^2}]^{1/2}}(3z^2r-r^3)\,dr\,dz \\
&= 2\pi\rho\int_{-a}^{a}\left|\frac{3z^2r^2}{2}-\frac{r^4}{4}\right|_0^{[b^2-\frac{b^2z^2}{a^2}]^{1/2}}dz \\
&= 2\pi\rho\int_{-a}^{a}\left\{\frac{3z^2}{2}\left[b^2-\frac{b^2z^2}{a^2}\right]-\frac{1}{4}\left[b^2-\frac{b^2z^2}{a^2}\right]^2\right\}dz \\
&= 2\pi\rho\int_{-a}^{a}\left\{z^4\left[-\frac{3b^2}{2a^2}-\frac{1}{4}\frac{b^4}{a^4}\right]+z^2\left[\frac{3b^2}{2}+\frac{b^4}{2a^2}\right]-\frac{b^4}{4}\right\}dz \\
&= 2\pi\rho\left\{\frac{2a^5}{5}\left[-\frac{3b^2}{2a^2}-\frac{1}{4}\frac{b^4}{a^4}\right]+\frac{2a^3}{3}\left[\frac{3b^2}{2}+\frac{b^4}{2a^2}\right]-\frac{ab^4}{2}\right\}
\end{aligned}$$

Problem 4.7

A localized distribution of charge has a charge density

$$\rho(\mathbf{r}) = \frac{1}{64\pi} r^2 e^{-r} \sin^2\theta.$$

(a) Make a multipole expansion of the potential due to this charge density and determine all the nonvanishing multipole moments. Write down the potential at large distances as a finite expansion in Legendre polynomials.

(b) Determine the potential explicitly at any point in space, and show that near the origin, correct to r^2 inclusive,

$$\Phi(\mathbf{r}) \approx \frac{1}{4\pi\epsilon_0}\left[\frac{1}{4} - \frac{r^2}{120}P_2(\cos\theta)\right].$$

(c) If there exists at the origin a nucleus with a quadrupole moment $Q = 10^{-28}$ m^2, determine the magnitude of the interaction energy, assuming that the unit of charge in $\rho(\mathbf{r})$ above is the electronic charge and the unit of length is the hydrogen Bohr radius $a_0 = 4\pi\epsilon_0\hbar^2/me^2 = 0.529\times 10^{-10}$ m. Express your answer as a frequency by dividing by Planck's constant h.

The charge density in the problem is that for the $m=\pm 1$ states of the $2p$ level in hydrogen, while the quadrupole interaction is of the same order as found in molecules.

(a) The multipole moments are given by

$$q_{lm} = \int \rho(r',\theta',\varphi')\, r'^l\, Y^*_{lm}(\theta',\varphi')\, dV'.$$

Strictly speaking, the multipole-expansion method is not valid for this problem, because ρ is nonvanishing everywhere; we can't talk of a "localized" charge distribution and the potential "far from it." However, the e^{-r} term causes the charge density to fall off so fast that beyond a certain distance there is effectively no charge, so we would expect that the multipole expansion would be valid at those distances.

Since there is no φ dependence in this problem, only moments with $m=0$ can be nonvanishing. For these we have

$$\begin{aligned} q_{l0} &= \frac{1}{64\pi}\cdot\sqrt{\frac{2l+1}{4\pi}}\cdot\int_0^\infty\int_0^\pi\int_0^{2\pi} r^{l+4}e^{-r}\sin^3\theta P_l(\cos\theta)\,dr\,d\theta\,d\varphi \\ &= \frac{1}{32}\sqrt{\frac{2l+1}{4\pi}}\cdot\left\{\int_0^\infty r^{l+4}e^{-r}\,dr\right\}\cdot\left\{\int_{-1}^1 (1-x^2)P_l(x)dx\right\} \qquad (9) \end{aligned}$$

The P_l integral is done easily:

$$\int_{-1}^{1}(1-x^2)P_l(x)\,dx = \int_{-1}^{1} P_l(x)\,dx - \int_{-1}^{1} x^2 P_l(x).$$

By the orthogonality of the Legendre polynomials, the first integral is only nonzero for $l = 0$, in which case it has the value 2. For the second integral we have Jackson's formula 3.32, from which we conclude that the integral has value 2/3 for $l = 0$, 4/15 for $l = 2$, and 0 otherwise. Hence

$$\int_{-1}^{1}(1-x^2)P_l(x)\,dx = \begin{cases} \frac{4}{3}, & l=0 \\ -\frac{4}{15} & l=2 \\ 0 & \text{otherwise.} \end{cases} \tag{10}$$

With this information, we only need to evaluate the r integral for the cases $l = 0$ and $l = 2$. With a few integrations by parts I derived the result

$$\int_0^\infty r^n e^{-r}\,dr = n!$$

so the r integral has value 24 and 720 for $l = 0$ and $l = 2$, respectively. (It seems a little weird to be getting dimensionless results for an integral that should have dimensions of $[\text{length}]^{n+1}$. The cause of this may be traced back to the appearance of the term e^{-r}. Obviously you can't raise e to a dimensionful power, so we have to understand r as being dimensionless here, which means that the length scale is hidden within the expressions. I don't usually like this since it makes it impossible to check results based on dimensional analysis...) Anyway, putting the results for the r and θ integrals into (9), we obtain

$$q_{00} = \sqrt{\frac{1}{4\pi}} \qquad q_{20} = -6\sqrt{\frac{5}{4\pi}}$$

The multipole expansion of the potential is

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\sum_{lm}\frac{4\pi}{2l+1}q_{lm}r^{-(l+1)}Y_l^m(\theta,\varphi)$$

With the multipole moments found above this is

$$\begin{aligned}\Phi(\mathbf{x}) &= \frac{1}{4\pi\epsilon_0}\left[\frac{1}{r} - \frac{6}{r^3}P_2(\cos\theta)\right] \\ &= \frac{1}{4\pi\epsilon_0 r}\left[1 - \frac{6}{r^2}P_2(\cos\theta)\right]\end{aligned}$$

(b) The explicit solution for the potential is obtained from the integral equation:

$$\begin{aligned}
\Phi(\mathbf{x}) &= \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}')}{|\mathbf{x}-\mathbf{x}'|}d\mathbf{x}' \\
&= \frac{1}{64\pi\epsilon_0}\sum_{lm}\frac{1}{2l+1}Y_{lm}(\theta,\varphi)\cdot\left\{\int \frac{r_<^l}{r_>^{l+1}}r'^4e^{-r'}\,dr'\right\}\cdot\left\{\int Y^*_{lm}(\theta',\varphi')\sin^3\theta' d\theta' d\varphi'\right\} \\
&= \frac{1}{128\pi\epsilon_0}\sum_{l}P_l(\cos\theta)\cdot\left\{\int \frac{r_<^l}{r_>^{l+1}}r'^4e^{-r'}\,dr'\right\}\cdot\left\{\int_{-1}^{1}(1-x^2)P_l(x)\,dx\right\}
\end{aligned}$$

where between the first and second lines we expanded $1/|\mathbf{x}-\mathbf{x}'|$ in terms of spherical harmonics. As above, terms with $m\neq 0$ must vanish from the sum. We have already done the θ integral, and plugging in the values we found for that integral we may write the potential as

$$\Phi(\mathbf{x}) = \frac{1}{128\pi\epsilon_0}\left[\frac{4}{3}I_0 - \frac{4}{15}I_2P_2(\cos\theta)\right] \tag{11}$$

where I_l refers to the r' integral above. The integrals must be split into two parts. For the $l=0$ case we have

$$\begin{aligned}
I_0 &= \frac{1}{r}\int_0^r r'^4e^{-r'}\,dr' + \int_r^\infty r'^3e^{-r'}\,dr' \\
&= \frac{1}{r}\left[24 - e^{-r}(r^3+6r^2+18r+24)\right] \\
&\longrightarrow 6 + O(x^5) \qquad \text{as } x\to 0
\end{aligned} \tag{12}$$

For the $l=2$ case we have

$$\begin{aligned}
I_2 &= \frac{1}{r^3}\int_0^r r'^6e^{-r'}\,dr' + r^2\int_r^\infty r'e^{-r'}\,dr' \\
&= \frac{1}{r^3}\left[720 - e^{-r}(x^6+6x^5+30x^4+120x^3+360x^4+720x+720)\right] + r^2\left[e^{-r}(1-r)\right] \\
&\longrightarrow r^2 - 2r^3 + \frac{3}{2}r^4 + O(x^5) \qquad \text{as } x\to 0
\end{aligned} \tag{13}$$

Plugging (12) and (13) into (11), we obtain

$$\Phi(x) = \frac{1}{4\pi\epsilon_0}\left[\frac{1}{4} - \frac{r^2}{120}P_2(\cos\theta) + O(r^3)\right]. \tag{14}$$

(c) The energy of the quadrupole interaction is given by the third term in Jackson's equation (4.24):

$$W = -\frac{1}{6}\sum_{ij}Q_{ij}\left.\frac{\partial E_j}{\partial x_i}\right|_{\mathbf{x}=0}$$

For an atomic nucleus, we have $-2Q_{11} = -2Q_{22} = Q_{33} = eQ$, with off-diagonal components of the tensor vanishing, so

$$\begin{aligned}
W &= \frac{eQ}{6}\left|\frac{1}{2}\frac{\partial E_x}{\partial x} + \frac{1}{2}\frac{\partial E_y}{\partial y} - \frac{\partial E_z}{\partial z}\right|_{\mathbf{x}=0} \\
&= \frac{eQ}{6}\left|\frac{1}{2}\nabla\cdot\mathbf{E} - \frac{3}{2}\frac{\partial E_z}{\partial z}\right|_{\mathbf{x}=0} \\
&= \frac{eQ}{6}\left|\frac{1}{2}\frac{\rho}{\epsilon_0} - \frac{3}{2}\frac{\partial E_z}{\partial z}\right|_{\mathbf{x}=0} \\
&= -\frac{eQ}{4}\left|\frac{\partial E_z}{\partial z}\right|_{\mathbf{x}=0} \\
&= \frac{eQ}{4}\left|\frac{\partial^2\Phi}{\partial z^2}\right|_{\mathbf{x}=0}
\end{aligned} \tag{15}$$

where, in going from the third-to-last to second-to-last lines we used the fact that the charge density in this problem vanishes at the origin. To compute the partial derivative of (14), it's easiest to rewrite that expression in cartesian coordinates. Ignoring the constant term, we have

$$\begin{aligned}
\Phi(\mathbf{x}) &= \frac{1}{480\pi\epsilon_0}r^2 P_2(\cos\theta) \\
&= \frac{1}{960\pi\epsilon_0}r^2(3\cos^2\theta - 1) \\
&= \frac{1}{960\pi\epsilon_0}(3z^2 - x^2 - y^2 - z^2) \\
&= \frac{1}{960\pi\epsilon_0}(2z^2 - x^2 - y^2).
\end{aligned}$$

Differentiating this, we have

$$\frac{\partial^2\Phi}{\partial z^2} = \frac{1}{240\pi\epsilon_0}$$

and putting this into (15) yields

$$W = \frac{eQ}{960\pi\epsilon_0}.$$

To make this dimensionally correct, we have to multiply by e on top and a_0^3 on the bottom. Doing this and putting in the numbers, we have

$$\begin{aligned}
\frac{W}{\hbar} &= \frac{1}{240} \cdot \frac{e^2 Q}{4\pi\epsilon_0 \hbar a_0^3} \\
&= \frac{1}{240} \cdot \frac{\alpha c Q}{a_0^3} \\
&= \frac{1}{240} \cdot \frac{1}{137} \cdot \frac{(3 \times 10^8 \text{ m} \cdot \text{s}^2)(10^{-28} \text{ m}^2)}{(0.529 \times 10^{-10} \text{ m})^3} \\
&= 6.16 \times 10^6 \text{ rad/s} \\
&\approx 1 \text{ MHz}.
\end{aligned}$$

This seems much too small to be true to me.

Solutions to Problems in Jackson, *Classical Electrodynamics*, Third Edition

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October 8, 2000

Chapter 4: Problems 8-13

Problem 4.8

A very long, right circular, cylindrical shell of dielectric constant ϵ/ϵ_0 and inner and outer radii a and b, respectively, is placed in a previously uniform electric field $\mathbf{E}_0$ with its axis perpendicular to the field. The medium inside and outside the cylinder has a dielectric constant of unity.

(a) Determine the potential and electric fields in the three regions, neglecting end effects.

(b) Sketch the lines of force for a typical case of $b \approx 2a$.

(c) Discuss the limiting forms of your solution appropriate for a solid dielectric cylinder in a uniform field, and a cylindrical cavity in a uniform dielectric.

We will take the axis of the cylinder to be the z axis and the electric field to be aligned with the x axis: $\mathbf{E}_0 = E_0\hat{\mathbf{i}}$. Since the cylinder is very long and we're told to neglect end effects, we can ignore the z direction altogether and treat this as a two-dimensional problem.

(a) The general solution of the Laplace equation in two dimensional polar coordinates is

$$\Phi(r,\varphi) = \sum [A_n r^n + B_n r^{-n}][C_n \sin(n\varphi) + D_n \cos(n\varphi)]$$

For the region inside the shell ($r < a$), the B coefficients must vanish to keep the potential from blowing up at the origin. Also, in the region outside the shell

($r > b$), the only positive power of r in the sum must be that which gives rise to the external electric field, i.e. $-E_0 r\cos\varphi$ with $A_n = 0$ for $n > 1$. With these observations we may write expressions for the potential in the three regions:

$$\Phi(r,\varphi) = \begin{cases} \sum r^n[A_n \sin n\varphi + B_n \cos n\varphi], & r < a \\ \sum r^n[C_n \sin n\varphi + D_n \cos n\varphi] + r^{-n}[E_n \sin n\varphi + F_n \cos n\varphi], & a < r < b \\ -E_0 r\cos\varphi + \sum r^{-n}[G_n \sin n\varphi + H_n \cos\varphi], & r > b \end{cases}$$

The normal boundary condition at $r = a$ is

$$\epsilon_0 \left.\frac{\partial\Phi}{\partial r}\right|_{x=a-} = \epsilon \left.\frac{\partial\Phi}{\partial r}\right|_{x=a+}$$

or

$$\frac{\epsilon_0}{\epsilon}\sum na^{n-1}[A_n \sin n\varphi + B_n \cos n\varphi] = \sum na^{n-1}[C_n \sin n\varphi + D_n \cos n\varphi] - na^{-(n+1)}[E_n \sin n\varphi + F_n \cos n\varphi]$$

From this we obtain two equations:

$$\frac{\epsilon_0}{\epsilon}A_n = C_n - E_n a^{-2n} \tag{1}$$

$$\frac{\epsilon_0}{\epsilon}B_n = D_n - F_n a^{-2n} \tag{2}$$

Next, the tangential boundary condition at $r = a$ is

$$\left.\frac{\partial\Phi}{\partial\varphi}\right|_{x=a+} = \left.\frac{\partial\Phi}{\partial\varphi}\right|_{x=a-}$$

or

$$\sum na^n[A_n \cos n\varphi - B_n \sin n\varphi] = \sum na^n[C_n \cos n\varphi - D_n \sin n\varphi] + na^{-n}[E_n \cos n\varphi - F_n \sin n\varphi]$$

from which we obtain two more equations:

$$A_n = C_n + E_n a^{-2n} \tag{3}$$

$$B_n = D_n + F_n a^{-2n} \tag{4}$$

Similarly, from the normal boundary condition at $r = b$ we obtain

$$-\frac{\epsilon_0}{\epsilon}E_0\cos\varphi - \frac{\epsilon_0}{\epsilon}\sum nb^{-(n+1)}[G_n \sin n\varphi + H_n \cos\varphi] = \sum nb^{n-1}[C_n \sin n\varphi + D_n \cos n\varphi] - nb^{-(n+1)}[E_n \sin n\varphi + F_n \cos\varphi]$$

which leads to

$$-\frac{\epsilon_0}{\epsilon}G_n = C_n b^{2n} - E_n \tag{5}$$

$$-\frac{\epsilon_0}{\epsilon}b^2 E_0 \delta_{n1} - \frac{\epsilon_0}{\epsilon}H_n = D_n b^{2n} - F_n \tag{6}$$

Finally, we have the tangential boundary condition at $r = b$:

$$bE_0 \sin\varphi + \sum nb^{-n}[G_n \cos n\varphi - H_n \sin n\varphi] = \\ \sum nb^n[C_n \cos n\varphi - D_n \sin n\varphi] + nb^{-n}[E_n \cos n\varphi - F_n \sin n\varphi]$$

giving

$$G_n = C_n b^{2n} + E_n \tag{7}$$

$$-b^2 E_0 \delta_{n1} + H_n = D_n b^{2n} + F_n. \tag{8}$$

The four equations (1), (3), (5), and (7) specify a degenerate system of linear equations, which can only be satisfied by taking $A_n = C_n = E_n = G_n = 0$ for all n. Next, for $n \neq 1$, the system of equations (2), (4), (6), and (8) specify the same degenerate system of equations, so $B_n = D_n = F_n = G_n = 0$ for $n \neq 0$. However, for $n = 1$, we have

$$\begin{aligned} \frac{\epsilon_0}{\epsilon}B_1 &= D_1 - F_1 a^{-2} & \qquad & D_1 = \frac{1}{2}\left(1 + \frac{\epsilon_0}{\epsilon}\right)B_1 \\ & & \Rightarrow & \\ B_1 &= D_1 + F_1 a^{-2} & & F_1 = \frac{1}{2}a^2\left(1 - \frac{\epsilon_0}{\epsilon}\right)B_1. \end{aligned}$$

and

$$\begin{aligned} -H_1 &= b^2 E_0 + \frac{\epsilon}{\epsilon_0}D_1 b^2 - \frac{\epsilon}{\epsilon_0}F_1 \\ H_1 &= b^2 E_0 + D_1 b^2 + F_1 \\ \rightarrow \qquad 0 &= 2b^2 E_0 + b^2\left(1 + \frac{\epsilon}{\epsilon_0}\right)D_1 + \left(1 - \frac{\epsilon}{\epsilon_0}\right)F_1 \end{aligned}$$

Substituting from above,

$$-4b^2 E_0 = \frac{1}{\epsilon\epsilon_0}\left[b^2(\epsilon + \epsilon_0)^2 - a^2(\epsilon - \epsilon_0)^2\right]B_1$$

or

$$B_1 = \frac{-4\epsilon\epsilon_0 b^2}{b^2(\epsilon + \epsilon_0)^2 - a^2(\epsilon - \epsilon_0)^2}E_0.$$

Then

$$D_1 = \frac{-2\epsilon_0(\epsilon+\epsilon_0)b^2}{b^2(\epsilon+\epsilon_0)^2 - a^2(\epsilon-\epsilon_0)^2}E_0$$

$$F_1 = \frac{-2\epsilon_0(\epsilon-\epsilon_0)a^2b^2}{b^2(\epsilon+\epsilon_0)^2 - a^2(\epsilon-\epsilon_0)^2}E_0$$

$$H_1 = \frac{-b^2(b^2-a^2)(\epsilon_0^2-\epsilon^2)}{b^2(\epsilon+\epsilon_0)^2 - a^2(\epsilon-\epsilon_0)^2}E_0.$$

The potential is

$$\Phi(r,\varphi) = \begin{cases} \dfrac{-4\epsilon\epsilon_0 b^2}{b^2(\epsilon+\epsilon_0)^2 - a^2(\epsilon-\epsilon_0)^2}\cdot E_0 r\cos\varphi, & r<a \\ \dfrac{-2\epsilon_0 b^2}{b^2(\epsilon+\epsilon_0)^2 - a^2(\epsilon-\epsilon_0)^2}\left((\epsilon+\epsilon_0)r + (\epsilon-\epsilon_0)\dfrac{a^2}{r}\right)E_0\cos\varphi, & a<r<b \\ \dfrac{-(b^2-a^2)(\epsilon_0^2-\epsilon^2)}{b^2(\epsilon+\epsilon_0)^2 - a^2(\epsilon-\epsilon_0)^2}\cdot\dfrac{b^2}{r}E_0\cos\varphi - E_0 r\cos\varphi, & b<r. \end{cases}$$

As $\epsilon \to \epsilon_0$, $\Phi \to -E_0 r\cos\varphi$ in all three regions, which is reassuring.

The electric field is

$$\mathbf{E}(r,\varphi) = \begin{cases} \dfrac{4\epsilon\epsilon_0 b^2}{b^2(\epsilon+\epsilon_0)^2 - a^2(\epsilon-\epsilon_0)^2}E_0\left[\cos\varphi\hat{\mathbf{r}} - \sin\varphi\hat{\varphi}\right], & r<a \\ \dfrac{2\epsilon_0 b^2}{b^2(\epsilon+\epsilon_0)^2 - a^2(\epsilon-\epsilon_0)^2}\left\{\left[(\epsilon+\epsilon_0) - (\epsilon-\epsilon_0)\dfrac{a^2}{r^2}\right]E_0\cos\varphi\hat{\mathbf{r}}\right. & \\ \qquad \left. -\left[(\epsilon+\epsilon_0) + (\epsilon-\epsilon_0)\dfrac{a^2}{r^2}\right]E_0\sin\varphi\hat{\varphi}\right\}, & a<r<b \\ -\dfrac{(b^2-a^2)(\epsilon_0^2-\epsilon^2)}{b^2(\epsilon+\epsilon_0)^2 - a^2(\epsilon-\epsilon_0)^2}\cdot\left(\dfrac{b}{r}\right)^2 E_0\left[\cos\varphi\hat{\mathbf{r}} + \sin\varphi\hat{\varphi}\right] & \\ \qquad +E_0\left[\cos\varphi\hat{\mathbf{r}} - \sin\varphi\hat{\varphi}\right], & b<r. \end{cases}$$

(b) In Figure 4.1 I've plotted the field lines for $b = 2a$, $\epsilon = 5\epsilon_0$. Also, as an appendix to this document I've included the C program I wrote to generate this plot.

(c) For a solid dielectric cylinder in a uniform field, we would have $a \to 0$. In that case the field would look like

$$\mathbf{E}(r,\varphi) = \begin{cases} \dfrac{2\epsilon_0}{\epsilon+\epsilon_0}E_0\hat{\mathbf{i}}, & r<b \\ E_0\hat{\mathbf{i}} - \dfrac{(\epsilon_0^2-\epsilon^2)}{(\epsilon+\epsilon_0)^2}\left(\dfrac{b}{r}\right)^2 E_0[\cos\varphi\hat{\mathbf{r}} + \sin\varphi\hat{\varphi}], & r>b \end{cases}$$

On the other hand, a cylindrical cavity in a uniform dielectric corresponds to

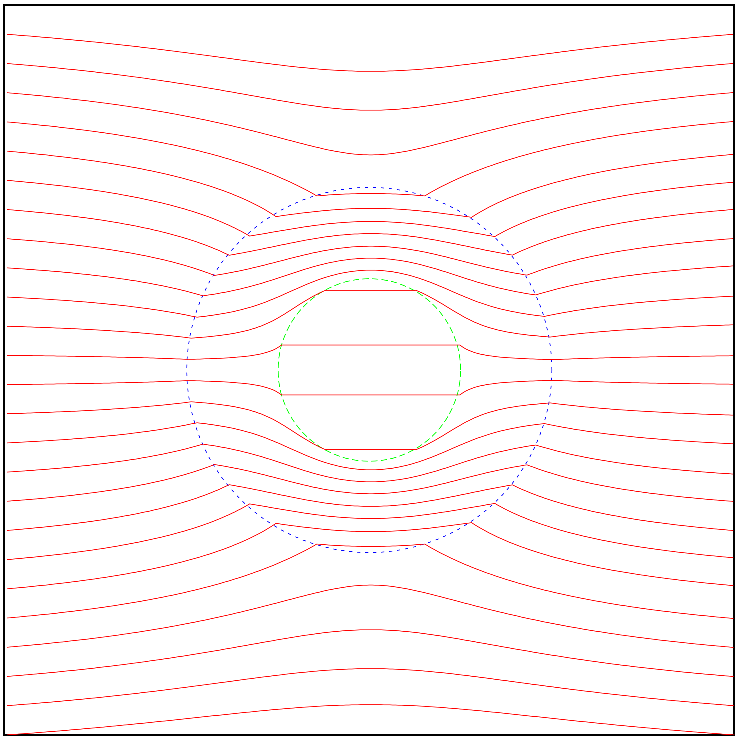

Figure 1: Field lines in Problem 4.8 for $b = 2a$, $\epsilon = 5\epsilon_0$.

$b \to \infty$, in which case the field becomes

$$\mathbf{E}(r,\varphi) = \begin{cases} \dfrac{4\epsilon\epsilon_0}{(\epsilon+\epsilon_0)^2} E_0 \hat{\mathbf{i}}, & r < a \\ \dfrac{2\epsilon_0}{(\epsilon+\epsilon_0)} E_0 \hat{\mathbf{i}} - \dfrac{2\epsilon_0(\epsilon-\epsilon_0)}{(\epsilon+\epsilon_0)^2}\left(\dfrac{a}{r}\right)^2 E_0[\cos\varphi\hat{\mathbf{r}} + \sin\varphi\hat{\varphi}], & r > a. \end{cases}$$

Problem 4.9

A point charge q is located in free space a distance d away from the center of a dielectric sphere of radius a ($a < d$) and dielectric constant ϵ/ϵ_0.

(a) Find the potential at all points in space as an expansion in spherical harmonics.

(b) Calculate the rectangular components of the electric field *near* the center of the sphere.

(c) Verify that, in the limit $\epsilon/\epsilon_0 \to \infty$, your result is the same as that for the conducting sphere.

We will take the origin of coordinates at the center of the sphere, and put the point charge on the z axis at $z = +h$. Then the problem has azimuthal symmetry.

(a) Since there is no free charge within the sphere, $\nabla\cdot\mathbf{D} = 0$ there. But since the permittivity is uniform within the sphere, we may also write $\nabla\cdot(\mathbf{D}/\epsilon) = \nabla\cdot\mathbf{E} = 0$ there. This means that polarization charge only exists on the surface of the sphere, so within the sphere the potential satisfies the normal Laplace equation, whence

$$\Phi(r,\theta) = \sum_l A_l r^l P_l(\cos\theta) \qquad (r < a).$$

Now, in the region $r > a$, the potential may be written as the sum of two components Φ_1 and Φ_2, where Φ_1 comes from the polarization charge on the surface of the sphere, while Φ_2 comes from the external point charge. Since Φ_1 satisfies the Laplace equation for $r > a$, we may expand it in Legendre polynomials:

$$\Phi_1(r,\theta) = \sum_l B_l r^{-(l+1)} P_l(\cos\theta) \qquad (r > a).$$

On the other hand, Φ_2 is just the potential due to a point charge at $z = d$:

$$\Phi_2(r,\theta) = \begin{cases} \dfrac{q}{4\pi\epsilon_0}\sum \dfrac{r^l}{d^{l+1}} P_l(\cos\theta), & r < d \\ \dfrac{q}{4\pi\epsilon_0}\sum \dfrac{d^l}{r^{l+1}} P_l(\cos\theta), & r > d. \end{cases} \tag{9}$$

Putting this all together we may write the potential in the three regions as

$$\Phi(r,\theta) = \begin{cases} \sum A_l r^l P_l(\cos\theta), & r < a \\ \sum \left[B_l r^{-(l+1)} + \dfrac{q}{4\pi\epsilon_0}\dfrac{r^l}{d^{l+1}} \right] P_l(\cos\theta), & a < r < d \\ \sum \left[B_l + \dfrac{q d^l}{4\pi\epsilon_0} \right] r^{-(l+1)} P_l(\cos\theta), & r > d. \end{cases}$$

The normal boundary condition at $r = a$ is

$$\begin{aligned}
\epsilon \left.\frac{\partial \Phi}{\partial r}\right|_{r=a-} &= \epsilon_0 \left.\frac{\partial \Phi}{\partial r}\right|_{r=a+} \\
\to \qquad \frac{\epsilon}{\epsilon_0} l A_l a^{l-1} &= -(l+1) B_l a^{-(l+2)} + \frac{l q a^{l-1}}{4\pi\epsilon_0 d^{l+1}} \\
\to \qquad A_l &= \frac{\epsilon_0}{\epsilon}\left[\frac{-(l+1)}{l} B_l a^{-(2l+1)} + \frac{q}{4\pi\epsilon_0 d^{l+1}}\right] \qquad (10)
\end{aligned}$$

The tangential boundary condition at $r = a$ is

$$\begin{aligned}
\left.\frac{\partial \Phi}{\partial \theta}\right|_{r=a-} &= \left.\frac{\partial \Phi}{\partial \theta}\right|_{r=a+} \\
\to \qquad A_l a^l &= B_l a^{-(l+1)} + \frac{q}{4\pi\epsilon_0}\frac{a^l}{d^{(l+1)}} \\
\to \qquad B_l &= A_l a^{2l+1} - \frac{q}{4\pi\epsilon_0}\frac{a^{2l+1}}{d^{l+1}} \qquad (11)
\end{aligned}$$

Combining (10) and (11), we obtain

$$\begin{aligned}
A_l &= \frac{1}{\frac{\epsilon}{\epsilon_0} + \frac{l+1}{l}}\left(\frac{2l+1}{l}\right)\frac{q}{4\pi\epsilon_0 d^{l+1}} \\
B_l &= \frac{1}{\frac{\epsilon}{\epsilon_0} + \frac{l+1}{l}}\left(1 - \frac{\epsilon}{\epsilon_0}\right)\frac{q a^{2l+1}}{4\pi\epsilon_0 d^{l+1}}
\end{aligned}$$

In particular, as $\epsilon/\epsilon_0 \to \infty$ we have

$$A_l \to 0$$

as must happen, since the field within a conducting sphere vanishes; and

$$B_l \to -\frac{q a^{2l+1}}{4\pi\epsilon_0 d^{l+1}}. \qquad (12)$$

With the coefficients (12), the potential outside the sphere due to the polarization charge at the sphere boundary is

$$\Phi_1(r,\theta) = \frac{1}{4\pi\epsilon_0}\left(-\frac{qa}{d}\right)\sum\left(\frac{a^2}{d}\right)^l \frac{1}{r^{l+1}} P_l(\cos\theta).$$

Comparing with (9) we see that this is just the potential of a charge $-qa/d$ on the z axis at $z = a^2/d$. This is just the size and position of the image charge we found in Chapter 2 for a point charge outside a conducting sphere.

(b) Near the origin, we have

$$\begin{aligned}\Phi(r,\theta) &= A_1 r P_1(\cos\theta) + A_2 r^2 P_2(\cos\theta) + \cdots \\ &= \frac{q}{4\pi\epsilon_0}\left[\frac{3\epsilon_0}{d^2(\epsilon+2\epsilon_0)} z + \frac{1}{2}\left(\frac{5\epsilon_0}{d^3(2\epsilon+3\epsilon_0)}\right)(z^2 - x^2 - y^2) + \cdots\right]\end{aligned}$$

so the field components are

$$\begin{aligned}E_x &= \frac{q}{4\pi\epsilon_0 d^2}\cdot\frac{5\epsilon_0}{2\epsilon+3\epsilon_0}\left(\frac{x}{d}\right) + \cdots \\ E_y &= \frac{q}{4\pi\epsilon_0 d^2}\cdot\frac{5\epsilon_0}{2\epsilon+3\epsilon_0}\left(\frac{y}{d}\right) + \cdots \\ E_z &= -\frac{q}{4\pi\epsilon_0 d^2}\left[\frac{3\epsilon_0}{\epsilon+2\epsilon_0} + \frac{5\epsilon_0}{2\epsilon+3\epsilon_0}\left(\frac{z}{d}\right) + \cdots\right]\end{aligned}$$

Problem 4.10

Two concentric conducting spheres of inner and outer radii a and b, respectively, carry charges $\pm Q$. The empty space between the spheres is half-filled by a hemispherical shell of dielectric (of dielectric constant ϵ/ϵ_0), as shown in the figure.

(a) Find the electric field everywhere between the spheres.

(b) Calculate the surface-charge distribution on the inner sphere.

(c) Calculate the polarization-charge density induced on the surface of the dielectric at $r = a$.

We'll orient this problem such that the boundary between the dielectric-filled space and the empty space is the xy plane. Then the region occupied by the dielectric is the region $a < r < b$, $0 < \theta < \pi/2$, and the problem has azimuthal symmetry.

(a) Since the dielectric has uniform permittivity, all the polarization charge exists on the boundary of the dielectric, so within its body we may take the potential to be a solution of the normal Laplace equation. The potential in the region between the spheres may then be written

$$\Phi(r,\theta) = \begin{cases} \sum [A_l r^l + B_l r^{-(l+1)}] P_l(\cos\theta), & 0 < \theta < \frac{\pi}{2} \\ \sum [C_l r^l + D_l r^{-(l+1)}] P_l(\cos\theta), & \frac{\pi}{2} < \theta < \pi \end{cases}$$

First let's apply the boundary conditions at the interface between the dielectric and free space. That region is described by $\theta = \pi/2$, $a < r < b$, and we must have

$$\epsilon \left. \frac{\partial \Phi}{\partial \theta} \right|_{\theta=\pi/2+} = \epsilon_0 \left. \frac{\partial \Phi}{\partial \theta} \right|_{\theta=\pi/2-}$$
$$\left. \frac{\partial \Phi}{\partial r} \right|_{\theta=\pi/2+} = \left. \frac{\partial \Phi}{\partial r} \right|_{\theta=\pi/2-}$$

which leads to

$$\sum \left[\frac{\epsilon}{\epsilon_0} A_l - C_l \right] P_l'(0) r^l + \left[\frac{\epsilon}{\epsilon_0} B_l - D_l \right] P_l'(0) r^{-l+1} = 0 \tag{13}$$
$$\sum l \left[A_l - C_l \right] P(0) r^{l-1} - (l+1) \left[B_l - D_l \right] P_l(0) r^{-l+2} = 0. \tag{14}$$

Since these equations must be satisfied for all r in the region $a < r < b$, the coefficients of each power of r must vanish identically. In (13), this requirement is automatically satisfied for l even, since $P_l'(0)$ vanishes for even l. Similarly, (14) is automatically satisfied for l odd. For other cases the vanishing of the coefficients must be brought about by taking

$$\frac{\epsilon}{\epsilon_0} A_l = C_l \qquad \frac{\epsilon}{\epsilon_0} B_l = D_l, \qquad l \text{ odd} \tag{15}$$
$$A_l = C_l \qquad B_l = D_l, \qquad l \text{ even.} \tag{16}$$

Next let's consider the charge at the surface of the inner sphere. There are actually two components of this charge; one component comes from the surface distribution of the free charge $+Q$ that exists on the sphere, and the other component comes from the bound polarization charge on the inner surface of the dielectric

Problem 4.13

Two long, coaxial, cylindrical conducting surfaces of radii a and b are lowered vertically into a liquid dielectric. If the liquid rises an average height h between the electrodes when a potential difference V is established between them, show that the susceptibility of the liquid is

$$\chi_e = \frac{(b^2 - a^2)\rho g h \ln(b/a)}{\epsilon_0 V^2}$$

where ρ is the density of the liquid, g is the acceleration due to gravity, and the susceptibility of air is neglected.

First let's work out what happens when a battery of fixed voltage V is connected between two coaxial conducting cylinders with simple vacuum between them. To begin, we can use Gauss' law to determine the **E** field between the cylinders. For our Gaussian pillbox we take a disk of thickness dz and radius r, $a < r < b$ centered on the axis of the cylinders. By symmetry there is no component of $\mathbf{E}$ normal to the top or bottom boundary surfaces, and the component normal to the side surfaces (the radial component) is uniform around the disc. Hence

$$\oint \mathbf{E} \cdot dA = 2\pi\, r\, dz E_\rho = \frac{q}{\epsilon_0} = \frac{1}{\epsilon_0}(2\pi\, a\, dz)\sigma$$

$$\to E_\rho(\rho) = \frac{a\sigma}{\epsilon_0 r}$$

where σ is the surface charge on the inner conductor. This must integrate to give the correct potential difference between the conductors:

$$V = -\int_a^b E_\rho(\rho) d\rho = -\frac{a\sigma}{\epsilon_0} \ln \frac{b}{a}$$

which tells us that, to establish a potential difference V between the conductors, the battery has to flow enough charge to establish a surface charge of magnitude

$$\sigma = \frac{\epsilon_0 V}{a \ln(b/a)} \tag{17}$$

on the cylinder faces (the surface charges are of opposite sign on the two cylinders).

It is useful to figure out the energy per unit length stored in the electric field between the cylinder plates here. This is just

$$\begin{aligned} W_v &= \frac{1}{2}\int_a^b \int_0^{2\pi} \mathbf{E} \cdot \mathbf{D}\, \rho\, d\rho\, d\phi \\ &= \pi\epsilon_0 \int_a^b E^2(\rho)\rho\, d\rho \\ &= \pi \frac{a^2\sigma^2}{\epsilon_0} \ln(b/a) \\ &= \frac{\pi\epsilon_0 V^2}{\ln(b/a)} \end{aligned} \tag{18}$$

where the v subscript stands for 'vacuum', since (18) is the energy per unit length stored in the field between the cylinders with just vacuum between them.

Now suppose we introduce a dielectric material between the cylinders. If the voltage between the cylinders is kept at V, then the $\mathbf{E}$ field must be just the same as it was in the no-dielectric case, because this field integrated from a to b must still give the same potential difference. However, in order to establish this same $\mathbf{E}$ field in the presence of the retarding effects of the dielectric, the battery now has to establish a surface charge that is *greater* that it was before by a factor (ϵ/ϵ_0). With this greater charge on the electrodes, the $\mathbf{D}$ field will now be bigger by a factor (ϵ/ϵ_0) than it was in our above calculation. So the energy per unit length stored in the field between the cylinders increases by a factor $(\epsilon/\epsilon_0 - 1)$ over the result (18):

$$\Delta W_d = (\epsilon - \epsilon_0)\frac{\pi V^2}{\ln(b/a)}.$$

On the other hand, to get to this point the battery has had to flow enough charge to increase the surface charges to be of magnitude (ϵ/ϵ_0) times greater than (17). In doing this the internal energy of the battery decreases by an amount equal to the work it had to do to flow the excess charge, namely

$$\Delta W_b = -V dQ = V(2\pi\, a\, d\sigma) = (\epsilon - \epsilon_0)\frac{2\pi V^2}{\ln(b/a)}$$

(per unit length). The energy lost by the battery is twice that gained by the dielectric, so the system with dielectric between the cylinders has lower overall energy than the system with vacuum between the cylinders by a factor

$$\Delta W = (\epsilon - \epsilon_0)\frac{\pi V^2}{\ln(b/a)} \tag{19}$$

(per unit length).

Turning now to the situation in this problem, we'll take the axis of the cylinders as the z axis, so that the surface of the liquid is parallel to the xy plane. We'll take the boundary between the liquid and the air above it to be at $z = 0$. With no potential between the cylinder plates, the liquid between the cylinders is at the same height as the liquid outside.

Now suppose a battery of fixed potential V is connected between the two cylinder plates. As we showed earlier, the combined system of battery and dielectric can lower its energy by having more of the dielectric rise up between the cylinders. However, at some point the energy win we get from this is balanced by the energy hit we take from the gravitational potential energy of having the excess liquid rise higher between the cylinders. The height at which we no longer gain by having more liquid between the cylinders is the height to which the system will settle.

So suppose that, with a battery keeping a voltage V between the electrodes, the liquid between the electrodes rises to a height h above the surface of the liquid outside the electrodes. The decrease in electrostatic energy this affords over the case with just vacuum filling that space is just (19) times the height, i.e.

$$E_e = -h(\epsilon - \epsilon_0)\frac{\pi V^2}{\ln(b/a)} \tag{20}$$

This must be balanced by the gravitational potential energy E_g of the excess liquid. E_g is easily calculated by noting that the area between the cylinders is $\pi(b^2 - a^2)$, so the mass of liquid contained in a height dh between the cylinders is $dm = \rho\pi(b^2 - a^2)dh$, and if this mass is at a height h above the liquid surface its excess gravitational energy is

$$dE_g = (dm)gh = \pi g\rho(b^2 - a^2)hdh.$$

Integrating over the excess height of liquid between the cylinders,

$$E_g = \pi g \rho (b^2 - a^2) \int_0^h h' \, dh' = \frac{1}{2} \pi g \rho (b^2 - a^2) h^2. \tag{21}$$

Comparing (20) to (21), we find that the gravitational penalty of the excess liquid just counterbalances the electrostatic energy reduction when

$$\begin{aligned} h &= \frac{2(\epsilon - \epsilon_0) V^2}{\rho g (b^2 - a^2) \ln(b/a)} \\ &= \frac{2 \chi_e \epsilon_0 V^2}{\rho g (b^2 - a^2) \ln(b/a)} \end{aligned}$$

Solving for χ_e,

$$\chi_e = \frac{\rho g h (b^2 - a^2) \ln(b/a)}{2 \epsilon_0 V^2}.$$

So I seem to be off by a factor of 2 somewhere.

Actually we should note one detail here. When the surface of the liquid between the cylinders rises, the surface of the liquid outside the cylinders must fall, since the total volume of the liquid is conserved. Hence there are really two other contributions to the energy shift, namely, the change in gravitational and electrostatic energies of the thin layer of liquid outside the cylinders that falls away when the liquid rises between the cylinders. But if the surface area of the vessel containing the liquid is sufficiently larger than the area between the cylinders, the difference layer will be thin and its energy shifts negligible.

Appendix

Source code for field line plotting program used in Problem 4.8.

```
/*
 * Program to draw field lines for Jackson problem 4.8.
 * Homer Reid   October 2000
 */

#include <stdio.h>
#include <math.h>
#include "/usr2/homer/include/GnuPlot.c"

#define EZ  1.0 /* permittivity of free space   */
#define EPS 5.0 /* permittivity of cylinder     */

#define E0  1.0        /* external field (irrelevant here) */

#define A 4.0 /* radius of inner cylinder     */
#define B 8.0 /* radius of outer cylinder     */

#define NUMLINES  25.0  /* number of field lines to draw */
#define NUMPOINTS 250.0  /* no. of pts to plot for each line */

#define DELTAX (4.0 * B) / NUMPOINTS    /* horiz spacing of pts */
#define DELTAY (4.0 * B) / NUMLINES     /* vert spacing of initial pts */

#define DENOM (B*B*(EPS+EZ)*(EPS+EZ) - A*A*(EPS-EZ)*(EPS-EZ))

/*
 * Return r component of electric field at position (r,phi).
 */
double Er(double r, double phi)
{
  double Coeff;

  if ( r < A )
   Coeff=(4.0*EPS*EZ*B*B)/DENOM;
  else if ( r < B )
   Coeff=(2*EPS*B*B/DENOM)*( (EPS+EZ) - (EPS-EZ)*(A*A)/(r*r) );
  else
   Coeff=1.0 - ((B*B - A*A)*(EZ*EZ-EPS*EPS)*(B*B)/(r*r*DENOM));

  return Coeff*E0*cos(phi);

}
```

```
/*
 * Return phi component of electric field at (r,phi).
 */
double Ephi(double r, double phi)
{
  double Coeff;

  if ( r < A )
   Coeff=(4.0*EPS*EZ*B*B)/DENOM;
  else if ( r < B )
   Coeff=(2*EPS*B*B/DENOM)*( (EPS+EZ) + (EPS-EZ)*(A*A)/(r*r) );
  else
   Coeff=1.0 + ((B*B - A*A)*(EZ*EZ-EPS*EPS)*(B*B)/(r*r*DENOM));

  return -Coeff*E0*sin(phi);

}

void main()
{
  double i,j,r,phi,x,y,dx,dy;
  double RComp,PhiComp;
  FILE *g;

  g=GnuPlot("Field lines");

  /*
   * Send basic GnuPlot configuration commands.
   */
  fprintf(g,"set terminal postscript portrait color\n");
  fprintf(g,"set output 'fig4.1.eps'\n");
  fprintf(g,"set multiplot \n");
  fprintf(g,"set size square\n");
  fprintf(g,"set noxtics\n");
  fprintf(g,"set noytics\n");
  fprintf(g,"set xrange [%g:%g]\n",-2.0*B,2.0*B);
  fprintf(g,"set yrange [%g:%g]\n",-2.0*B,2.0*B);

  /*
   * Draw circles at r=a and r=b.
   */
  fprintf(g,"plot '-' t '', '-' t '' with lines, '-' t '' with lines\n");
  fprintf(g,"e\n");
  for(phi=0; phi<=2*M_PI; phi+=(2*M_PI/100))
   fprintf(g,"%g %g\n",A*cos(phi),A*sin(phi));
```

```
fprintf(g,"e\n");
for(phi=0; phi<=2*M_PI; phi+=(2*M_PI/100))
 fprintf(g,"%g %g\n",B*cos(phi),B*sin(phi));
fprintf(g,"e\n");

/*
 * Draw field lines.
 */
for (i=1.0; i<=NUMLINES; i+=1.0)
 {
   /*
    * Compute starting x and y coordinates and initiate plot.
    */
   x=-2.0*B; y=2.0*B * ((NUMLINES - 2.0*i)/NUMLINES);
   fprintf(g,"plot '-' t '' with lines\n");

   /*
    * Plot NUMPOINTS points for this field line.
    */
   for (j=0.0; j<NUMPOINTS; j+=1.0)
    {
      /*
       * compute polar coordinates of present location
       */
      r=sqrt(x*x + y*y);
      if (x==0.0)
       phi=(y>0.0) ? M_PI/2.0 : -M_PI/2.0;
      else
       phi=atan(y/x);

      /*
       * compute rise and run of electric field
       */
      RComp=Er(r,phi);
      PhiComp=Ephi(r,phi);
      dx=cos(phi)*RComp - sin(phi)*PhiComp;
      dy=sin(phi)*RComp + cos(phi)*PhiComp;

      /*
       * bump x coordinate forward a fixed amount, and y
       * coordinate up or down by an amount depending on
       * the direction of the electric field at this point
       */
      x+=DELTAX;
      y+=DELTAX * (dy/dx);
      fprintf(g,"%g %g\n",x,y);
```

```
      };
     fprintf(g,"e\n");
   };

  printf("Thank you for your support.\n");

}
```

Solutions to Problems in Jackson, *Classical Electrodynamics*, Third Edition

Homer Reid

November 8, 2000

Chapter 5: Problems 1-10

Problem 5.1

Starting with the differential expression

$$d\mathbf{B} = \frac{\mu_0 I}{4\pi} d\mathbf{l}' \times \frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|^3}$$

for the magnetic induction at the point P with coordinate $\mathbf{x}$ produced by an increment of current $I\,d\mathbf{l}'$ at $\mathbf{x}'$, show explicitly that for a closed loop carrying a current I the magnetic induction at P is

$$\mathbf{B} = \frac{\mu_0 I}{4\pi} \nabla\Omega$$

where Ω is the solid angle subtended by the loop at the point P. This corresponds to a magnetic scalar potential, $\Phi_M = -\mu_0 I\Omega/4\pi$. The sign convention for the solid angle is that Ω is positive if the point P views the "inner" side of the surface spanning the loop, that is, if a unit normal $\mathbf{n}$ to the surface is defined by the direction of current flow via the right-hand rule, Ω is positive if $\mathbf{n}$ points *away* from the point P, and negative otherwise. This is the same convention as in Section 1.6 for the electric dipole layer.

I like to change the notation slightly: the observation point is $\mathbf{r}_1$, the coordinate of a point on the current loop is $\mathbf{r}_2$, and the displacement vector (pointing to the observation point) is $\mathbf{r}_{12} = \mathbf{r}_1 - \mathbf{r}_2$.

The solid angle subtended by the current loop at $\mathbf{r}_1$ is given by a surface integral over the loop:

$$\Omega = \int_S \frac{\cos\gamma\, dA}{r_{12}^2}$$

Solutions to Problems in Jackson, *Classical Electrodynamics*, Third Edition

Homer Reid

February 11, 2001

Chapter 5: Problems 10-18

Problem 5.10

A circular current loop of radius a carrying a current I lies in the $x - y$ plane with its center at the origin.

(a) Show that the only nonvanishing component of the vector potential is

$$A_\phi(\rho, z) = \frac{\mu_0 I a}{\pi} \int_0^\infty dk \cos kz \, I_1(k\rho_<) K_1(k\rho_>)$$

where $\rho_<(\rho_>)$ is the smaller (larger) of a and ρ.

(b) Show that an alternative expression for A_ϕ is

$$A_\phi(\rho, z) = \frac{\mu_0 I a}{2} \int_0^\infty dk e^{-k|z|} J_1(ka) J_1(k\rho).$$

(c) Write down integral expressions for the components of magnetic induction, using the expressions of parts a and b. Evaluate explicitly the components of $\mathbf{B}$ on the z axis by performing the necessary integrations.

(a) Translating Jackson's equation (5.33) into cylindrical coordinates, we have

$$J_\phi = I\delta(z)\delta(\rho - a) \tag{1}$$

Following Jackson, we take the observation point $\mathbf{x}$ on the x axis, so its coordinates are $(\rho, \phi = 0, z)$. Since there is no current in the z direction, and since the current density is cylindrically symmetric, there is no vector potential in the ρ or z directions. In the ϕ direction we have

$$\begin{aligned}
A_\phi &= -A_x \sin\phi + A_y \cos\phi = A_y \\
&= \frac{\mu_0}{4\pi}\int \frac{J_y(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\,d\mathbf{x}' \\
&= \frac{\mu_0}{4\pi}\int \frac{J_\phi(\mathbf{x}')\cos\phi'}{|\mathbf{x}-\mathbf{x}'|}\,d\mathbf{x}' \\
&= \frac{\mu_0}{4\pi}\,\mathrm{Re}\int \frac{J_\phi(\mathbf{x}')e^{i\phi'}}{|\mathbf{x}-\mathbf{x}'|}\,d\mathbf{x}' \\
&= \frac{\mu_0}{4\pi}\,\mathrm{Re}\int J_\phi(\mathbf{x}')e^{i\phi'}\left[\frac{2}{\pi}\sum_{m=-\infty}^{\infty}\int_0^\infty e^{im(\phi-\phi')}\cos[k(z-z')]I_m(k\rho_<)K_m(k\rho_>)\,dk\right]d\mathbf{x}'
\end{aligned}$$

where we substituted in Jackson's equation (3.148). Rearranging the order of integration and remembering that $\phi = 0$, we have

$$A_\phi = \frac{\mu_0}{2\pi^2}\,\mathrm{Re}\sum_{m=-\infty}^{\infty}\int_0^\infty\left[\int J_\phi(\mathbf{x}')e^{i(1-m)\phi'}\cos[k(z-z')]I_m(k\rho_<)K_m(k\rho_>)d\mathbf{x}'\right]dk$$

If $m = 1$, the ϕ integral yields 2π; otherwise it vanishes. Thus

$$A_\phi = \frac{\mu_0}{\pi}\int_0^\infty\left[\int_0^\infty\int_{-\infty}^{\infty} J_\phi(r',z')\cos[k(z-z')]I_1(k\rho_<)K_1(k\rho_>)\rho'\,dz'\,dr'\right]dk$$

Substituting (1), we have

$$A_\phi = \frac{Ia\mu_0}{\pi}\int_0^\infty \cos kz\, I_1(k\rho_<)K_1(k\rho_>)\,dk.$$

(b) The procedure for obtaining this expression is identical to the one I just went through, but with the expression from Problem 3.16(b) used for the Green's function instead of equation (3.148).

(c) Let's suppose that the observation point is in the interior region of the current loop, so $\rho_< = \rho, \rho_> = a$. Then

$$\begin{aligned}
B_\rho &= [\nabla\times\mathbf{A}]_\rho = -\frac{\partial A_\phi}{\partial z} \\
&= -\frac{Ia\mu_0}{\pi}\int_0^\infty k\,\sin kz\, I_1(k\rho)K_1(ka)\,dk \\
B_z &= [\nabla\times\mathbf{A}]_z = \frac{1}{\rho}A_\phi + \frac{\partial A_\phi}{\partial\rho} \\
&= \frac{Ia\mu_0}{\pi}\int_0^\infty \cos kz\left[\frac{I_1(k\rho)}{\rho} + kI_1'(k\rho)\right]K_1(ka)\,dk
\end{aligned}$$

As $\rho = 0$, $I_1(\rho) \to 0$, $I_1(\rho)/\rho \to 1/2$, and $I_1'(\rho) \to 1/2$, so

$$\begin{aligned} B_\rho(\rho = 0) &= 0 \\ B_z(\rho = 0) &= \frac{Ia\mu_0}{\pi} \int_0^\infty k \cos kz K_1(ka)\, dk \\ &= \frac{Ia\mu_0}{\pi} \frac{\partial}{\partial z} \int_0^\infty \sin kz K_1(ka) dk \end{aligned}$$

The integral may be done by parts:

$$\int_0^\infty \sin kz K_1(kz)\, dk = \left| -\frac{1}{a} \sin kz K_0(ka) \right|_0^\infty + \frac{z}{a} \int_0^\infty \cos kz K_0(ka)\, dk$$

K_0 is finite at zero but sin vanishes there, and sin is finite at infinity but K_0 vanishes there, so the first term vanishes. The integral in the second term is Jackson's equation (3.150). Plugging it in to the above,

$$\begin{aligned} B_z(\rho = 0) &= \frac{I\mu_0}{2} \frac{\partial}{\partial z} \frac{z}{(z^2 + a^2)^{1/2}} \\ &= \frac{I\mu_0}{2} \frac{a^2}{(z^2 + a^2)^{3/2}}. \end{aligned}$$

Problem 5.11

A circular loop of wire carrying a current I is located with its center at the origin of coordinates and the normal to its plane having spherical angles θ_0, ϕ_0. There is an applied magnetic field, $B_x = B_0(1 + \beta y)$ and $B_y = B_0(1 + \beta x)$.

(a) Calculate the force acting on the loop without making any approximations. Compare your result with the approximate result (5.69). Comment.

(b) Calculate the torque in lowest order. Can you deduce anything about the higher order contributions? Do they vanish for the circular loop? What about for other shapes?

(a) Basically we're dealing with two different reference frames here. In the "lab" frame, $\mathcal{R}$, the magnetic field exists only in the xy plane, and the normal to the current loop has angles θ_0, ϕ_0. We define the "rotated" frame $\mathcal{R}'$ by aligning the z' axis with the normal to the current loop, so that in $\mathcal{R}'$ the current loop exists only in the $x'y'$ plane, but the magnetic field now has a z' component.

The force on the current loop is

$$\mathbf{F} = \int (\mathbf{J} \times \mathbf{B}) dV. \tag{2}$$

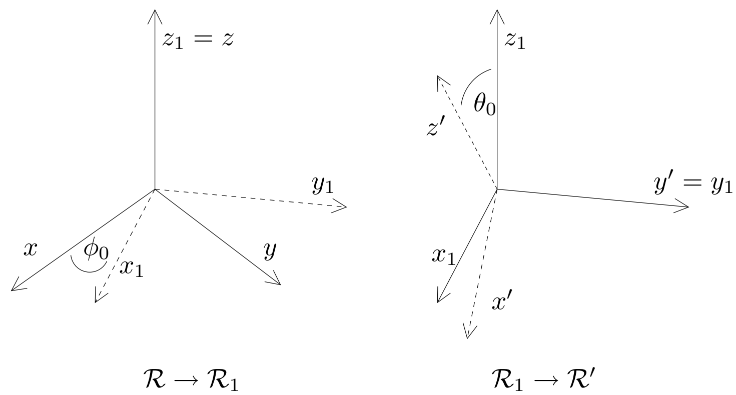

Figure 1: Successive coordinate transformations in Problem 5.11.

The components of $\mathbf{J}$ are easy to express in $\mathcal{R}'$, but more complicated in $\mathcal{R}$; the opposite is true for $\mathbf{B}$. There are two ways to do the problem: we can work out the components of $\mathbf{J}$ in $\mathcal{R}$ and do the integral in $\mathcal{R}$, or we can work out the components of $\mathbf{B}$ in $\mathcal{R}'$ and do the integral in $\mathcal{R}'$, in which case we would have to transform the components of the force back to $\mathcal{R}$ to get the answer we desire. I think the former approach is easier.

To derive the transformation matrix relating the coordinates of a point in $\mathcal{R}$ and $\mathcal{R}'$, I imagined that the transformation arose from two separate transformations, as depicted in figure (**??**). The first transformation is a rotation through ϕ_0 around the z axis, which takes us from $\mathcal{R}$ to an intermediate frame $\mathcal{R}_1$. Then we rotate through θ_0 around the y_1 axis, which takes us to $\mathcal{R}'$. Evidently, the coordinates of a point in the various frames are related by

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} \cos\phi_0 & \sin\phi_0 & 0 \\ -\sin\phi_0 & \cos\phi_0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{3}$$

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\theta_0 & 0 & -\sin\theta_0 \\ 0 & 1 & 0 \\ \sin\theta_0 & 0 & \cos\theta_0 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} \tag{4}$$

Multiplying matrices,

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\theta_0\cos\phi_0 & \cos\theta_0\sin\phi_0 & -\sin\theta_0 \\ -\sin\phi_0 & \cos\phi_0 & 0 \\ \sin\theta_0\cos\phi_0 & \sin\theta_0\sin\phi_0 & \cos\theta_0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}. \tag{5}$$

This matrix also gives us the transformation between unit vectors in the two frames:

$$\begin{pmatrix} \hat{\mathbf{i}}' \\ \hat{\mathbf{j}}' \\ \hat{\mathbf{k}}' \end{pmatrix} = \begin{pmatrix} \cos\theta_0\cos\phi_0 & \cos\theta_0\sin\phi_0 & -\sin\theta_0 \\ -\sin\phi_0 & \cos\phi_0 & 0 \\ \sin\theta_0\cos\phi_0 & \sin\theta_0\sin\phi_0 & \cos\theta_0 \end{pmatrix} \begin{pmatrix} \hat{\mathbf{i}} \\ \hat{\mathbf{j}} \\ \hat{\mathbf{k}} \end{pmatrix}. \tag{6}$$

We will also the inverse transformation, i.e. the expressions for coordinates in $\mathcal{R}$ in terms of coordinates in $\mathcal{R}'$:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \cos\theta_0\cos\phi_0 & -\sin\phi_0 & \sin\theta_0\cos\phi_0 \\ \cos\theta_0\sin\phi_0 & \cos\phi_0 & \sin\theta_0\sin\phi_0 \\ -\sin\theta_0 & 0 & \cos\theta_0 \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}. \tag{7}$$

To do the integral in (2) it's convenient to parameterize a point on the current loop by an angle ϕ' reckoned from the x' axis in $\mathcal{R}'$. If the loop radius is a, then the coordinates of a point on the loop are $x' = a\cos\phi'$, $y' = a\sin\phi'$, and the current density/volume element product is

$$\begin{aligned}
\mathbf{J}\,dV &= I\,d\,l = (Ia\,d\phi')\hat{\phi}' \\
&= Ia\,d\phi'[-\sin\phi'\hat{\mathbf{i}}' + \cos\phi'\hat{\mathbf{j}}'] \\
&= Ia\,d\phi'\Big[(-\sin\phi'\cos\theta_0\cos\phi_0 - \cos\phi'\sin\phi_0)\hat{\mathbf{i}} \\
&\qquad + (\sin\phi'\sin\phi_0 + \cos\phi'\cos\phi_0)\hat{\mathbf{j}} + (\sin\phi'\sin\theta_0)\hat{\mathbf{k}}\Big]
\end{aligned}$$

We also need the components of the $\mathbf{B}$ field at a point on the current loop:

$$\begin{aligned}
\mathbf{B}(\phi') &= B_0[1+\beta y(\phi')]\hat{\mathbf{i}} + B_0[1+\beta x(\phi')] \\
&= B_0[1 + a\beta(\cos\phi'\cos\theta_0\sin\phi_0 + \sin\phi'\cos\phi_0)]\hat{\mathbf{i}} + B_0[1 + a\beta(\cos\phi'\cos\theta_0\cos\phi_0 - \sin\phi'\sin\phi_0)]\hat{\mathbf{j}}
\end{aligned}$$

The components of the cross product are

$$\begin{aligned}
[\mathbf{J}\times\mathbf{B}]_x\,dV &= -J_zB_y\,dV \\
&= (\cdots)\beta Ia^2B_0 d\phi'\big(\sin^2\phi'\sin\theta_0\sin\phi_0\big) \\
[\mathbf{J}\times\mathbf{B}]_y\,dV &= J_zB_x\,dV \\
&= (\cdots) + \beta Ia^2B_0\,d\phi'\big(\sin^2\phi'\sin\theta_0\cos\phi_0\big) \\
[\mathbf{J}\times\mathbf{B}]_z\,dV &= (J_xB_y - J_yB_x)\,dV \\
&= (\cdots) + 0
\end{aligned}$$

where we only wrote out terms containing a factor of $\cos^2\phi'$ or $\sin^2\phi'$, since only these terms survive after the integral around the current loop (we grouped all the remaining terms into $(\cdots)$). In the surviving terms, $\cos^2\phi'$ and $\sin^2\phi'$ turn into factors of π after the integral around the loop. Then the force components are

$$\begin{aligned}
F_x &= \pi\beta Ia^2B_0\sin\theta_0\sin\phi_0 \\
F_y &= \pi\beta Ia^2B_0\sin\theta_0\cos\phi_0 \\
F_z &= 0.
\end{aligned}$$

To compare this with the first-order approximate result, note that the magnetic moment has magnitude $\pi a^2 I$ and is oriented along the z' axis:

$$\mathbf{m} = \pi a^2 I\hat{\mathbf{k}}' = \pi a^2 I\Big(\sin\theta_0\cos\phi_0\hat{\mathbf{i}} + \sin\theta_0\sin\phi_0\hat{\mathbf{j}} + \cos\theta_0\hat{\mathbf{k}}\Big)$$

so

$$\begin{aligned}\nabla\big(\mathbf{B}\cdot\mathbf{m}\big) &= \nabla\big(B_0(1+\beta y)m_x + B_0(1+\beta x)m_y\big)\\ &= B_0\beta\big(m_y\hat{\mathbf{i}} + m_x\hat{\mathbf{j}}\big)\\ &= \pi\beta I a^2 B_0\big(\sin\theta_0\sin\phi_0\hat{\mathbf{i}} + \sin\theta_0\cos\phi_0\hat{\mathbf{j}}\big)\end{aligned}$$

in exact agreement with the result we calculated so laboriously above.

Problem 5.12

Two concentric circular loops of radii a, b and currents I, I', respectively ($b < a$), have an angle α between their planes. Show that the torque on one of the loops is about the line of intersection of the two planes containing the loops and has the magnitude

$$N = \frac{\mu_0\pi I I' b^2}{2a}\sum_{n=0}^{\infty}\frac{(n+1)}{(2n+1)}\left[\frac{\Gamma(n+3/2)}{\Gamma(n+2)\Gamma(3/2)}\right]^2\left(\frac{b}{a}\right)^{2n} P^1_{2n+1}(\cos\alpha).$$

The torque on the smaller loop is

$$\begin{aligned}\mathbf{N} &= \int \mathbf{r}\times\Big(\mathbf{J}_b(\mathbf{r})\times\mathbf{B}_a(\mathbf{r})\Big)d\mathbf{r}\\ &= \int\Big\{\big[\mathbf{r}\cdot\mathbf{B}_a(\mathbf{r})\big]\mathbf{J}_b(\mathbf{r}) - \big[\mathbf{r}\cdot\mathbf{J}_b(\mathbf{r})\big]\mathbf{B}_a(\mathbf{r})\Big\}d\mathbf{r}.\end{aligned}$$

where J_b is the current density of the smaller loop and B_a is the magnetic field of the larger loop. But $\mathbf{r}\cdot\mathbf{J}_b$ vanishes, because the current flows in a circle around the origin—there is no current flowing toward or away from the origin. Thus

$$\mathbf{N} = \int rB_r(\mathbf{r})\mathbf{J}_b(\mathbf{r})d\mathbf{r} \tag{8}$$

where B_r is the radial component of the magnetic field of the larger current loop.

As in the last problem, it's convenient to define two reference frames for this situation. Let $\mathcal{R}$ be the frame in which the smaller loop (radius b, current I) lies in the xy plane, and $\mathcal{R}'$ the frame in which the larger loop lies in the $x'y'$ plane. We might as well take the line of intersection of the two planes to be the y axis, so $y = y'$. Then the z' axis has spherical coordinates ($\theta = \alpha, \phi = 0$) in

$\mathcal{R}$, and for transforming back and forth between the two frames we may use the transformation matrices we derived in the last problem, with $\theta_0 = \alpha$, $\phi_0 = 0$. If we choose to evaluate the integral (8) in frame $\mathcal{R}$, the current density is

$$\mathbf{J}_b(\mathbf{r}) = I\delta(r-b)\delta(\theta-\pi/2)\big[-\sin\phi\hat{\mathbf{i}} + \cos\phi\hat{\mathbf{j}}\big]$$

so the components of the torque are

$$N_x = -Ib^2\int_0^{2\pi} B_r(r=b,\theta=\pi/2,\phi)\sin\phi\,d\phi \tag{9}$$

$$N_y = Ib^2\int_0^{2\pi} B_r(r=b,\theta=\pi/2,\phi)\cos\phi\,d\phi \tag{10}$$

To do the integral in (8), we need an expression for the radial component B_r of the field of the larger loop. Of course, we already have an expression for the field in $\mathcal{R}'$: in that frame the field is just that of a circular current loop in the $x'y'$ plane, Jackson's equation (5.48):

$$B'_r(r',\theta') = \frac{\mu_0 I' a}{2r'}\sum_{l=0}^{\infty}\frac{(-1)^l(2l+1)!!}{2^l l!}\frac{r_<^{2l+1}}{r_>^{2l+2}}P_{2l+1}(\cos\theta').$$

We are interested in evaluating this field at points along the smaller current loop, and for all such points $r=b$; then $r_< = b$, $r_> = a$ and we have

$$B'_r(r'=b,\theta') = \frac{\mu_0 I'}{2a}\sum_{l=0}^{\infty}\frac{(-1)^l(2l+1)!!}{2^l l!}\left(\frac{b}{a}\right)^{2l}P_{2l+1}(\cos\theta'). \tag{11}$$

To transform this to frame $\mathcal{R}$, we first note that, since the origins of $\mathcal{R}$ and $\mathcal{R}'$ coincide, the unit vectors $\hat{\mathbf{r}}$ and $\hat{\mathbf{r}}'$ coincide, so $B_r = B'_r$. Next, (11) expresses the field in terms of $\cos\theta'$, the polar angle in frame $\mathcal{R}'$. How do we write this in terms of the angles θ and ϕ in frame $\mathcal{R}$? Well, note that

$$\begin{aligned}
\cos\theta' &= \frac{z'}{r}\\
&= \frac{x\sin\alpha + z\cos\alpha}{r}\\
&= \frac{r\sin\theta\cos\phi\sin\alpha + r\cos\theta\cos\alpha}{r}\\
&= \sin\theta\sin\alpha\cos\phi + \cos\theta\cos\alpha
\end{aligned} \tag{12}$$

where in the second line we used the transformation matrix from Problem 5.11 to write down z' in terms of x and z. Equation (12) is telling us what our coordinates in $\mathcal{R}'$ are in terms of our coordinates in $\mathcal{R}$; if a point has angular coordinates θ,ϕ in $\mathcal{R}$, then (12) tells us what angle θ' it has in $\mathcal{R}'$. (We could also work out what the azimuthal angle ϕ' would be, but we don't need to, because (11) doesn't depend on ϕ'.)

To express the Legendre function in (11) with the argument (12), we may make use of the addition theorem for associated Legendre polynomials:

$$\begin{aligned} P_l(\cos\theta') &= P_l(\cos\theta\cos\alpha + \sin\theta\sin\alpha\cos\phi) \\ &= P_l(\cos\theta)P_l(\cos\alpha) + 2\sum_{m=1}^{l} P_l^m(\cos\theta)P_l^m(\cos\alpha)\cos m\phi. \end{aligned}$$

Of course, the smaller loop exists in the xy plane, so for all points on that loop we have $\theta = \pi/2$, whence

$$P_l(\cos\theta') = P_l(0)P_l(\cos\alpha) + 2\sum_{m=1}^{l} P_l^m(0)P_l^m(\cos\theta)\cos m\phi.$$

We may now write down an expression for the radial component of the magnetic field of the larger loop, evaluated at points on the smaller loop, in terms of the angle ϕ that goes from 0 to 2π around that loop:

$$\begin{aligned} B_r(\phi) = \frac{\mu_0 I'}{2a}\sum_{l=0}^{\infty}\frac{(-1)^l(2l+1)!!}{2^l l!}\left(\frac{b}{a}\right)^{2l}\Bigg\{ & P_{2l+1}(0)P_{2l+1}(\cos\alpha) \\ & + 2\sum_{m=1}^{2l+1} P_{2l+1}^m(0)P_{2l+1}^m(\cos\alpha)\cos m\phi\Bigg\}. \end{aligned}$$

This looks ugly, but in fact when we plug it into the integrals (9) and (10) the $\sin\phi$ and $\cos\phi$ terms beat against the $\cos m\phi$ term, integrating to 0 in the former case and $\pi\delta_{m1}$ in the latter. The torque is

$$\begin{aligned} N_x &= 0 \\ N_y &= \frac{\pi\mu_0 I I' b^2}{a}\sum_{l=0}^{\infty}\frac{(-1)^l(2l+1)!!}{2^l l!}\left(\frac{b}{a}\right)^{2l} P_{2l+1}^1(0)P_{2l+1}^1(\cos\alpha). \end{aligned}$$

To finish we just need to rewrite the numerical factor under the sum:

$$\begin{aligned} \frac{(-1)^l(2l+1)!!}{2^l l!}P_{2l+1}^1(0) &= \frac{(2l+1)!!}{2^l l!}\left[\frac{\Gamma(l+3/2)}{\Gamma(l+1)\Gamma(3/2)}\right] \\ &= \frac{(2l+3-2)(2l+3-4)(2l+3-6)\cdots(5)(3)}{2^l\Gamma(l+1)}\left[\frac{\Gamma(l+3/2)}{\Gamma(l+1)\Gamma(3/2)}\right] \\ &= \frac{(l+3/2-1)(l+3/2-2)\cdots(5/2)(3/2)}{\Gamma(l+1)}\left[\frac{\Gamma(l+3/2)}{\Gamma(l+1)\Gamma(3/2)}\right] \\ &= \left[\frac{\Gamma(l+3/2)}{\Gamma(l+1)\Gamma(3/2)}\right]^2 \\ &= (l+1)^2\left[\frac{\Gamma(l+3/2)}{\Gamma(l+2)\Gamma(3/2)}\right]^2 \end{aligned}$$

So my answer is

$$N_y = \frac{\pi\mu_0 I I' b^2}{a}\sum_{l=0}^{\infty}(l+1)^2\left[\frac{\Gamma(l+3/2)}{\Gamma(l+2)\Gamma(3/2)}\right]^2\left(\frac{b}{a}\right)^{2l} P^1_{2l+1}(\cos\alpha).$$

Evidently I'm off by a factor of $1/(l+1)(2l+1)$ under the sum, but I can't find where. Can anybody help?

Problem 5.13

> A sphere of radius a carries a uniform surface-charge distribution σ. The sphere is rotated about a diameter with constant angular velocity ω. Find the vector potential and magnetic-flux density both inside and outside the sphere.

Problem 5.14

> A long, hollow, right circular cylinder of inner (outer) radius a (b), and of relative permeability μ_r, is placed in a region of initially uniform magnetic-flux density $\mathbf{B}_0$ at right angles to the field. Find the flux density at all points in space, and sketch the logarithm of the ratio of the magnitudes of $\mathbf{B}$ on the cylinder axis to $\mathbf{B}_0$ as a function of $\log_{10}\mu_r$ for $a^2/b^2 = 0.5, 0.1$. Neglect end effects.

We'll take the cylinder axis as the z axis of our coordinate system, and we'll take $\mathbf{B}_0$ along the x axis: $\mathbf{B}_0 = B_0\hat{\mathbf{i}}$. To the extent that we ignore end effects, we may imagine the fields to have no z dependence, so we effectively have a two dimensional problem.

There are two distinct current distributions in this problem. The first is a current distribution $\mathbf{J}_{\text{free}}$ giving rise to the uniform field $\mathbf{B}_0$ far away from the cylinder; this current distribution is only nonvanishing at points outside the cylinder. The second is a current distribution $\mathbf{J}_{\text{bound}} = \nabla\times\mathbf{M}$ existing only within the cylinder. Since there is no free current within the cylinder or in its inner region, the equations determining $\mathbf{H}$ in those regions are

$$\nabla\cdot\mathbf{B} = \nabla\cdot(\mu\mathbf{H}) = 0, \qquad \nabla\times\mathbf{H} = \mathbf{J}_{\text{free}} = 0.$$

These imply that, within the cylinder and in its inner region, we may derive $\mathbf{H}$ from a scalar potential: $\mathbf{H} = -\nabla\Phi_m$, with Φ_m satisfying the Laplace equation.

In the external region, there is free current, so things are not so simple. To proceed we may separate the $\mathbf{H}$ field in the external region into two components: one that arises from the free current, and one that arises from the bound currents within the cylinder. The former is just $(1/\mu_0)\mathbf{B}_0$ and the second is again derivable from a scalar potential satisfying the Laplace equation. So, in the external region, $\mathbf{H} = (1/\mu_0)\mathbf{B}_0 - \nabla\Phi_m$.

So our task is to find expressions for Φ_m in the three regions such that the boundary conditions on $\mathbf{B}$ and $\mathbf{H}$ are satisfied at the borders of the regions. Writing down the solutions of the 2-D Laplace equation in the three regions, and excluding terms which blow up as $\rho \to 0$ or $\rho \to \infty$, we have

$$\Phi_m(\rho,\phi) = \begin{cases} \sum_{n=1}^{\infty} \rho^n \big[A_n \cos n\phi + B_n \sin n\phi\big] & r < a \\ \sum_{n=1}^{\infty} \Big\{\rho^n \big[C_n \cos n\phi + D_n \sin n\phi\big] + \rho^{-n}\big[E_n \cos n\phi + F_n \sin n\phi\big]\Big\} & a < r < b \\ \sum_{n=1}^{\infty} \rho^{-n} \big[G_n \cos n\phi + H_n \sin n\phi\big] & r > b \end{cases}$$

Actually, we may argue on symmetry grounds that the sin terms must all vanish: otherwise, the fields would take different values on the positive and negative y axes, but there is nothing in the problem distinguishing these axes from each other. With this simplification we may write down expressions for the components of the $\mathbf{H}$ field in the three regions:

$$H_r = \begin{cases} -\dfrac{\partial}{\partial r}\Phi_m = \displaystyle\sum_{n=1}^{\infty} -nA_n\rho^{n-1}\cos n\phi, & r < a \\ -\dfrac{\partial}{\partial r}\Phi_m = \displaystyle\sum_{n=1}^{\infty} -n\Big(C_n\rho^{n-1} - E_n\rho^{-(n+1)}\Big)\cos n\phi, & a < r < b \\ (1/\mu_0)B_{0r} - \dfrac{\partial}{\partial r}\Phi_m = \Big[(1/\mu_0)B_0\cos\phi + \displaystyle\sum_{n=1}^{\infty} nG_n\rho^{-(n+1)}\cos n\phi\Big], & r < b. \end{cases}$$

$$H_\phi = \begin{cases} -\dfrac{\partial}{\partial \phi}\Phi_m = \displaystyle\sum_{n=1}^{\infty} nA_n\rho^{n-1}\sin n\phi, & r < a \\ -\dfrac{\partial}{\partial \phi}\Phi_m = \displaystyle\sum_{n=1}^{\infty} n\Big(C_n\rho^{n-1} + E_n\rho^{-(n+1)}\Big)\sin n\phi, & a < r < b \\ (1/\mu_0)B_{0\phi} - \dfrac{\partial}{\partial \phi}\Phi_m = \Big[-(1/\mu_0)B_0\sin\phi + \displaystyle\sum_{n=1}^{\infty} nG_n\rho^{-(n+1)}\sin n\phi\Big], & r < b. \end{cases}$$

The boundary conditions at $r = b$ are that μH_ρ and H_ϕ be continuous, where $\mu = \mu_0$ outside the cylinder and $\mu_r\mu_0$ inside. With the above expressions for the components of $\mathbf{H}$, we have

$$\frac{1}{\mu_0}B_0\cos\phi + \sum_{n=1}^{\infty} nG_n b^{-(n+1)}\cos n\phi = \mu_r\sum_{n=1}^{\infty} -n\Big(C_n b^{n-1} - E_n b^{-(n+1)}\Big)\cos n\phi$$

$$-\frac{1}{\mu_0}B_0\sin\phi + \sum_{n=1}^{\infty} nG_n b^{-(n+1)}\sin n\phi = \sum_{n=1}^{\infty} n\Big(C_n b^{n-1} + E_n b^{-(n+1)}\Big)\sin n\phi.$$

We may multiply both sides of these by $\cos n\phi$ and $\sin n\phi$ and integrate from

0 to 2π to find

$$\frac{1}{\mu_0}B_0 + G_1 b^{-2} = -\mu_r C_1 + \mu_r E_1 b^{-2} \tag{13}$$

$$G_n b^{-(n+1)} = -\mu_r\Big(C_n b^{n-1} - E_n b^{-(n-1)}\Big), \qquad n \neq 1 \tag{14}$$

$$-\frac{1}{\mu_0}B_0 + G_1 b^{-2} = C_1 + E_1 b^{-2} \tag{15}$$

$$G_n b^{-(n+1)} = \Big(C_n b^{n-1} + E_n b^{-(n+1)}\Big), \qquad n \neq 1 \tag{16}$$

Similarly, at $r = a$ we obtain

$$A_1 = \mu_r C_1 - \mu_r E_1 a^{-2} \tag{17}$$

$$A_n a^{n-1} = \mu_r\Big(C_n a^{n-1} - E_n a^{-(n+1)}\Big), \qquad n \neq 1$$

$$A_1 = C_1 + E_1 a^{-2} \tag{18}$$

$$A_n a^{n-1} = \Big(C_n a^{n-1} + E_n a^{-(n+1)}\Big), \qquad n \neq 1. \tag{19}$$

For $n \neq 1$, the only solution turns out to be $A_n = C_n = E_n = G_n = 0$. For $n = 1$, multiplying (15) by μ_r and adding and subtracting with (13) yields

$$2\mu_r C_1 = -(\mu_r + 1)\frac{B_0}{\mu_0} + (\mu_r - 1)G_1 b^{-2} \tag{20}$$

$$2\mu_r E_1 = (1 - \mu_r)\frac{B_0}{\mu_0}b^2 + (\mu_r + 1)G_1. \tag{21}$$

On the other hand, multiplying (18) by μ_r and adding and subtracting with (17) yields

$$2\mu_r C_1 = (\mu_r + 1)A_1 \tag{22}$$

$$2\mu_r E_1 = (\mu_r - 1)a^2 A_1. \tag{23}$$

Equating (20) with (22), we find

$$A_1 = -\frac{B_0}{\mu_0} + \frac{(\mu_r - 1)}{(\mu_r + 1)}G_1 b^{-2}$$

while equating (21) with (23) yields

$$A_1 = -\frac{B_0}{\mu_0}\left(\frac{b^2}{a^2}\right) + \frac{(\mu_r + 1)}{(\mu_r - 1)}G_1 a^{-2}$$

and now equating *these* two equations gives

$$G_1 = \left[1 - \left(\frac{a}{b}\right)^2\right] \frac{(\mu_r^2 - 1)b^2}{(\mu_r + 1)^2 b^2 - (\mu_r - 1)^2 a^2}\left(\frac{B_0}{\mu_0}\right) b^2.$$

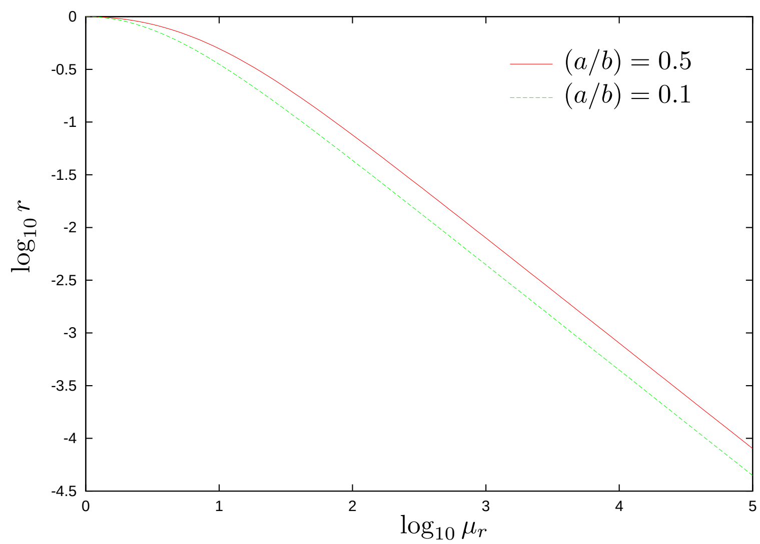

Figure 2: Damping of field inside cylindrical cylinder of permeability μ_r.

The other coefficients may be worked out from this one:

$$A_1 = \frac{-4\mu_r b^2}{(\mu_r+1)^2 b^2 - (\mu_r-1)^2 a^2}\frac{B_0}{\mu_0}$$

$$C_1 = \frac{-2(\mu_r+1) b^2}{(\mu_r+1)^2 b^2 - (\mu_r-1)^2 a^2}\frac{B_0}{\mu_0}$$

$$E_1 = \frac{-2(\mu_r-1) b^2}{(\mu_r+1)^2 b^2 - (\mu_r-1)^2 a^2}\frac{B_0}{\mu_0}a^2.$$

The **H** field is

$$\begin{aligned}\mathbf{H} &= \frac{4\mu_r b^2}{(\mu_r+1)^2 b^2 - (\mu_r-1)^2 a^2}\frac{B_0}{\mu_0}\hat{\mathbf{i}}, & r < a\\ &= \frac{2b^2}{(\mu_r+1)^2 b^2 - (\mu_r-1)^2 a^2}\frac{B_0}{\mu_0}\Big\{\Big[(\mu_r+1) + (\mu_r-1)\left(\frac{a}{r}\right)^2\Big]\hat{\mathbf{i}} - 2(\mu_r-1)\left(\frac{a}{r}\right)^2\cos\phi\hat{\mathbf{r}}\Big\}, & a < r < b\\ &= \frac{B_0}{\mu}\hat{\mathbf{i}} + \frac{(b^2-a^2)(\mu_r^2-1)}{(\mu_r+1)^2 b^2 - (\mu_r-1)^2 a^2}\left(\frac{B_0}{\mu_0}\right)\left(\frac{b^2}{r^2}\right)\big[\hat{\mathbf{i}} + 2\sin\phi\,\hat{\phi}\big], & r > b.\end{aligned}$$

The ratio r of the field within the cylinder to the external field is

$$r = \frac{4\mu_r}{(\mu_r+1)^2 - (\mu_r-1)^2\frac{a^2}{b^2}}.$$

This relationship is graphed in Figure

Problem 5.16

A circular loop of wire of radius a and negligible thickness carries a current I. The loop is centered in a spherical cavity of radius $b > a$ in a large block of soft iron. Assume that the relative permeability of the iron is effectively infinite and that of the medium in the cavity, unity.

(a) In the approximation of $b \gg a$, show that the magnetic field at the center of the loop is augmented by a factor $(1 + a^3/2b^3)$ by the presence of the iron.

(b) What is the radius of the "image" current loop (carrying the same current) that simulates the effect of the iron for $r < b$?

(a) There are two distinct current distributions in this problem: the free current density $\mathbf{J}_1$ flowing in the loop, and the bound current density $\mathbf{J}_2$ flowing in the iron. These give rise to two fields $\mathbf{B}_1$ and $\mathbf{B}_2$, which must be summed at each point in space to get the observed field.

$\mathbf{B}_1$ is just the field of a planar current loop, which Jackson has already worked out for us in his section 5.5:

$$\mathbf{B}_{1r} = \begin{cases} \dfrac{\mu_0 I}{2a} \displaystyle\sum_{n=0}^{\infty} \dfrac{(-1)^n (2n+1)!!}{2^n n!} \left(\dfrac{r}{a}\right)^{2n} P_{2n+1}(\cos\theta), & r < a \\ \dfrac{\mu_0 I a^2}{2r^3} \displaystyle\sum_{n=0}^{\infty} \dfrac{(-1)^n (2n+1)!!}{2^n n!} \left(\dfrac{a}{r}\right)^{2n} P_{2n+1}(\cos\theta), & r > a. \end{cases} \tag{24}$$

$$\mathbf{B}_{1\theta} = \begin{cases} \dfrac{\mu_0 I}{4a} \displaystyle\sum_{n=0}^{\infty} \dfrac{(-1)^n (2n-1)!!}{2^{n-1} n!} \left(\dfrac{r}{a}\right)^{2n} P^1_{2n+1}(\cos\theta), & r < a \\ -\dfrac{\mu_0 I a^2}{4r^3} \displaystyle\sum_{n=0}^{\infty} \dfrac{(-1)^n (2n+1)!!}{2^n (n+1)!} \left(\dfrac{a}{r}\right)^{2n} P^1_{2n+1}(\cos\theta), & r > a. \end{cases} \tag{25}$$

On the other hand, since $\mathbf{J}_2$ vanishes for $r < b$, the field $\mathbf{B}_2$ to which it gives rise has no divergence or curl in that region, which means that throughout the region it may be derived from a scalar potential satisfying the Laplace equation:

$$\mathbf{B}_2 = -\nabla\Phi_m = -\nabla\left[\sum_{n=0}^{\infty} A_n r^n P_n(\cos\theta)\right]$$

$$\rightarrow B_{2r} = \sum_{n=1}^{\infty} n A_n r^{n-1} P_n(\cos\theta) \tag{26}$$

$$B_{2\theta} = \sum_{n=1}^{\infty} A_n r^{n-1} P^1_n(\cos\theta) \tag{27}$$

Since the iron filling the space $r > b$ is assumed to have infinite permeability, the **H** field (and hence the **B** field, since $\mathbf{B} = \mathbf{H}$ for $r < b$) must be strictly radial at the boundary $r = b$. The A_n coefficients are thus determined by the requirement that (27) and (25) sum to zero at $r = b$:

$$\sum_{n=1}^{\infty} A_n b^{n-1} P_n^1(\cos\theta) = \frac{\mu_0 I a^2}{4b^3} \sum_{n=0}^{\infty} \frac{(-1)^n (2n+1)!!}{2^n (n+1)!} \left(\frac{a}{b}\right)^{2n} P_{2n+1}^1(\cos\theta).$$

The orthogonality of the associated Legendre polynomials requires that each term in the sum cancel individually, whence

$$A_{2n} = 0$$

$$A_{2n+1} = \frac{\mu_0 I a^2}{4b^3} \frac{(-1)^n (2n+1)!!}{2^n (n+1)!} \left(\frac{a}{b^2}\right)^{2n}.$$

Then the field of the bound current in the iron is determined everywhere in the region $r < b$:

$$B_{2r} = \frac{\mu_0 I a^2}{4b^3} \sum_{n=0}^{\infty} \frac{(-1)^n (2n+1)(2n+1)!!}{2^n (n+1)!} \left(\frac{ar}{b^2}\right)^{2n} P_{2n+1}(\cos\theta) \qquad (28)$$

$$B_{2\theta} = \frac{\mu_0 I a^2}{4b^3} \sum_{n=0}^{\infty} \frac{(-1)^n (2n+1)!!}{2^n (n+1)!} \left(\frac{ar}{b^2}\right)^{2n} P_{2n+1}^1(\cos\theta). \qquad (29)$$

As $r \to 0$, $B_{2\theta} \to 0$ and $B_{2r} \to \mu_0 I a^2 / 4b^3$, while $B_{1r} \to \mu_0 I / 2a$, so the total field at $r = 0$ is

$$B_r(r=0) = B_{1r}(r=0) + B_{2r}(r=0) = \frac{\mu_0 I}{2a} + \frac{\mu_0 I a^2}{4b^3} = \frac{\mu_0 I}{2a}\left[1 + \frac{a^3}{2b^3}\right].$$

(b) The $\mathbf{B}_2$ field may be attributed to an image current ring outside $r = b$ if, for suitable redefinitions of I and a, the expressions (28) and (29) can be made to look like the $r < a$ versions of (24) and (25).

Problem 5.18

A circular loop of wire having a radius a and carrying a current I is located in vacuum with its center a distance d away from a semi-infinite slab of permeability μ. Find the force acting on the loop when

(a) the plane of the loop is parallel to the face of the slab,

(b) the plane of the loop is perpendicular to the face of the slab.

(c) Determine the limiting form of your answer to parts a and b when $d \gg a$. Can you obtain these limiting values in some simple and direct way?

(a) We'll take the loop to be at $z = +d$, and the slab of permeability μ to occupy the space $z < 0$, so that the boundary surface is $z = 0$.

In the region $z < 0$, there is no free current, so $\nabla \times \mathbf{H} = 0$ everywhere; thus $\mathbf{H}$ may be obtained from a scalar potential, $\mathbf{H} = -\nabla\Phi_m$, and since $\nabla \cdot \mathbf{H} = 0$ as well we have $\nabla^2\Phi_m = 0$. The azimuthally symmetric solution of the Laplace equation in cylindrical coordinates that remains finite as $z \to -\infty$ is

$$\Phi_m(z < 0) = \int_0^\infty dk\, A(k) e^{kz} J_0(k\rho), \tag{30}$$

and from this we obtain

$$\begin{aligned} H_\rho(z<0) &= -\frac{\partial}{\partial\rho}\Phi_m = -\int_0^\infty dk\, kA(k)e^{kz}J_0'(k\rho) \\ &= \int_0^\infty dk\, kA(k)e^{kz}J_1(k\rho) & (31) \\ H_z(z<0) &= -\frac{\partial}{\partial z}\Phi_m = -\int_0^\infty dk\, kA(k)e^{kz}J_0(k\rho). & (32) \end{aligned}$$

On the other hand, for $z > 0$ we may decompose the $\mathbf{H}$ field into two components: one component $\mathbf{H}_1$ arising from the current loop, and a second component $\mathbf{H}_2$ arising from the bound currents running in the slab. $\mathbf{H}_1$ is just given by the curl of the vector potential we worked out in Problem 5.10:

$$\mathbf{H}_1 = \frac{1}{\mu_0}\nabla\times\mathbf{A}, \qquad \mathbf{A} = A_\phi\hat{\phi}, \qquad A_\phi = \begin{cases} \dfrac{\mu_0 I a}{2}\displaystyle\int_0^\infty dk\, e^{-k(z-d)}J_1(ka)J_1(k\rho), & z > d \\ \dfrac{\mu_0 I a}{2}\displaystyle\int_0^\infty dk\, e^{-k(d-z)}J_1(ka)J_1(k\rho), & z < d. \end{cases}$$

so

$$\begin{aligned} H_{1\rho} &= -\frac{1}{\mu_0}\frac{\partial}{\partial z}A_\phi \\ &= \begin{cases} \dfrac{Ia}{2}\displaystyle\int_0^\infty dk\, ke^{-k(z-d)}J_1(ka)J_1(k\rho), & z > d \\ -\dfrac{Ia}{2}\displaystyle\int_0^\infty dk\, ke^{-k(d-z)}J_1(ka)J_1(k\rho), & z < d. \end{cases} \end{aligned} \tag{33}$$

$$\begin{aligned} H_{1z} &= \frac{1}{\mu_0}\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_\phi) \\ &= \begin{cases} \dfrac{Ia}{2}\displaystyle\int_0^\infty dk\, ke^{-k(z-d)}J_1(ka)\left[\frac{1}{k\rho}J_1(k\rho) - J_0(k\rho)\right] & z > d \\ \dfrac{Ia}{2}\displaystyle\int_0^\infty dk\, ke^{-k(d-z)}J_1(ka)\left[\frac{1}{k\rho}J_1(k\rho) - J_0(k\rho)\right], & z < d. \end{cases} \end{aligned} \tag{34}$$

In the last two equations we may use Jackson's identity (3.87),

$$\frac{1}{k\rho}J_1(k\rho) = \frac{1}{2}\left[J_0(k\rho) + J_2(k\rho)\right]$$

to rewrite H_{1z} as

$$H_{1z} = \begin{cases} \frac{Ia}{4}\int_0^\infty dk\, e^{-k(z-d)} J_1(ka)\left[J_2(k\rho) - J_0(k\rho)\right], & z > d \\ \frac{Ia}{4}\int_0^\infty dk\, e^{-k(d-z)} J_1(ka)\left[J_2(k\rho) - J_0(k\rho)\right], & z < d. \end{cases} \tag{35}$$

Since the $\mathbf{H}_2$ field arises entirely from bound currents, it may also be derived from a scalar potential Φ_m satisfying the Laplace equation. The azimuthally symmetric solution of the Laplace equation in cylindrical coordinates that remains finite for all ρ and as $z \to +\infty$ is

$$\Phi_m(z > 0) = \int_0^\infty dk\, B(k) e^{-kz} J_0(k\rho)$$

and the components of $\mathbf{H}_2$ are

$$H_{2r}(z > 0) = -\int_0^\infty dk\, kB(k) e^{-kz} J_1(k\rho) \tag{36}$$

$$H_{2z}(z > 0) = \int_0^\infty dk\, kB(k) e^{-kz} J_0(k\rho). \tag{37}$$

The required forms of the functions $A(k)$ and $B(k)$ are determined by the boundary conditions on $\mathbf{H}$ at the medium boundary, $z = 0$:

$$H_\rho(z = 0_-) = H_\rho(z = 0_+) \qquad \mu H_\rho(z = 0_-) = \mu_0 H_\rho(z = 0_+).$$

Equating (32) with the sum of (**??**) and (**??**), we have

$$-\int_0^\infty dk\, kA(k) J_0(k\rho) = \frac{\mu_0 Ia}{2}\int_0^\infty dk\, ke^{-kd} J_1(ka)\left(J_2(k\rho) - J_0(k\rho)\right) + \int_0^\infty dk\, kB(k) J_0(k\rho)$$

Solutions to Problems in Jackson, *Classical Electrodynamics*, Third Edition

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April 20, 2001

Chapter 5: Problems 19-27

Problem 5.19

A magnetically "hard" material is in the shape of a right circular cylinder of length L and radius a. The cylinder has a permanent magnetization M_0, uniform throughout its volume and parallel to its axis.

(a) Determing the magnetic field $\mathbf{H}$ and magnetic induction $\mathbf{B}$ at all points on the axis of the cylinder, both inside and outside.

(b) Plot the ratios $\mathbf{B}/\mu_0 M_0$ and $\mathbf{H}/M_0$ at all points on the axis of the cylinder, both inside and outside.

There is no free current in this problem, so $\mathbf{H}(\rho, z)$ may be derived from a scalar potential $\Phi_m(\rho, z)$ satisfying the Laplace equation. Dividing space into three regions

$$\Phi_m = \begin{cases} \displaystyle\int_0^\infty dk\, A(k) e^{-kz} J_0(k\rho), & z > L/2 \\ \displaystyle\int_0^\infty dk\, \left[B(k)e^{kz} + C(k)e^{-kz}\right] J_0(k\rho), & -L/2 < z < L/2 \\ \displaystyle\int_0^\infty dk\, D(k) e^{kz} J_0(k\rho), & z < -L/2. \end{cases}$$

The tangential boundary condition at $z = +L/2$ is

$$\left.\frac{\partial \Phi_m}{\partial \rho}\right|_{z=\frac{L}{2}+} = \left.\frac{\partial \Phi_m}{\partial \rho}\right|_{z=\frac{L}{2}-}$$
$$\Rightarrow \int_0^\infty dk\, k A(k) e^{-kL/2} J_1(k\rho) = \int_0^\infty dk\, k \left[B(k) e^{kL/2} + C(k) e^{-kL/2} \right] J_1(k\rho) \tag{1}$$

This must hold for all ρ. Multiplying both sides by $\rho J_1(k'\rho)$, integrating from $\rho = 0$ to $\rho = \infty$, and using the identity

$$\int_0^\infty d\rho\, \rho J_n(k\rho) J_n(k'\rho) = \frac{1}{k}\delta(k - k') \tag{2}$$

we obtain from (1) the relation

$$A(k) = B(k) e^{kL} + C(k). \tag{3}$$

The perpendicular boundary condition at $z = +L/2$ is

$$B_z(z = L/2+) = B_z(L/2-)$$

or

$$\mu_0 H_z(z = L/2+) = \mu_0 \big[H_z(z = L/2-) + M_z(z = L/2-) \big]$$
$$\left.\frac{\partial \Phi_m}{\partial z}\right|_{z=\frac{L}{2}+} = \left.\frac{\partial \Phi_m}{\partial z}\right|_{z=\frac{L}{2}-} + M(\rho)$$
$$\Rightarrow \int_0^\infty dk\, k A(k) e^{-kL/2} J_0(k\rho) = \int_0^\infty dk\, k \left[-B(k) e^{kL/2} + C(k) e^{-kL/2} \right] J_0(k\rho) + M(\rho) \tag{4}$$

where

$$M(\rho) = \begin{cases} M_1, & \rho < a \\ 0, & \rho > a. \end{cases}$$

Now we multiply both sides of (4) by $\rho J_0(k'\rho)$ and integrate from $\rho = 0$ to $\rho = \infty$ to obtain

$$\begin{aligned} A(k) &= -B(k) e^{kL} + C(k) + M_1 e^{kL/2} \int_0^a \rho J_0(k\rho) d\rho \\ &= -B(k) e^{kL} + C(k) + \gamma(k) \end{aligned} \tag{5}$$

where we defined

$$\gamma(k) = M_1 e^{kL/2} \int_0^a \rho J_0(k\rho) d\rho = \frac{a M_1}{k} e^{kL/2} J_1(ka).$$

The solution of eqs. (3) and (5) is

$$B(k) = \frac{1}{2}e^{-kL}\gamma(k) \qquad C(k) = A(k) - \frac{1}{2}\gamma(k). \tag{6}$$

From the boundary conditions at $z = -L/2$ we may similarly obtain the relations

$$\begin{aligned} B(k) + C(k)e^{kL} &= D(k) \\ B(k) - C(k)e^{kL} &= D(k) - \gamma(k) \end{aligned}$$

which may be solved to yield

$$B(k) = D(k) - \frac{1}{2}\gamma(k) \qquad C(k) = \frac{1}{2}e^{-kL}\gamma(k). \tag{7}$$

Comparing (6) and (7) we find

$$\begin{aligned} A(k) = D(k) &= \frac{M_1 a}{k}\cosh\frac{kL}{2}J_1(ka) \\ B(k) = C(k) &= \frac{M_1 a}{2k}e^{-kL/2}J_1(ka). \end{aligned}$$

Then the components of the **H** field are

$$H_\rho = \begin{cases} M_1 a \int_0^\infty dk\, \cosh\frac{kL}{2}e^{-kz}J_1(ka)J_1(k\rho), & z > L/2 \\ M_1 a \int_0^\infty dk\, e^{-kL/2}\cosh(kz)J_1(ka)J_1(k\rho), & -L/2 < z < L/2 \\ M_1 a \int_0^\infty dk\, \cosh\frac{kL}{2}e^{kz}J_1(ka)J_1(k\rho), & z < -L/2 \end{cases}$$

$$H_z = \begin{cases} M_1 a \int_0^\infty dk\, \cosh\frac{kL}{2}e^{-kz}J_1(ka)J_0(k\rho), & z > L/2 \\ -M_1 a \int_0^\infty dk\, e^{-kL/2}\sinh(kz)J_1(ka)J_0(k\rho), & -L/2 < z < L/2 \\ -M_1 a \int_0^\infty dk\, \cosh\frac{kL}{2}e^{kz}J_1(ka)J_0(k\rho), & z < -L/2. \end{cases}$$

Problem 5.23

A right circular cylinder of length L and radius a has a uniform lengthwise magnetization M.

(a) Show that, when it is placed with its flat end against an infinitely permeable plane surface, it adheres with a force

$$F = 2\mu_0 a L M^2 \left[\frac{K(k) - E(k)}{k} - \frac{K(k_1) - E(k_1)}{k_1} \right]$$

where

$$k = \frac{2a}{\sqrt{4a^2 + L^2}}, \qquad k_1 = \frac{a}{\sqrt{a^2 + L^2}}.$$

(b) Find the limiting form of the force if $L \gg a$.

We'll define our coordinate system so that the z axis is the cylinder axis, and we'll take the surface of the permeable medium at $z = 0$.

Our general strategy for this problem will be as follows. First, we'll find the magnetic field $\mathbf{H}_0$ that exists in all space when the cylinder is pressed up flat against the infinitely permeable medium. Then we'll calculate the shift dE in the energy of the magnetic field incurred by moving the cylinder up a small distance dz off the surface of the medium. The force on the cylinder is then readily calculated as $F = -dE/dz$.

To calculate the energy shift incurred by moving the cylinder a distance dz away from the permeable medium, we won't have to go through and completely recalculate the fields and their energy in the new configuration. Instead, we can use the following little trick. When we move the cylinder up a distance dz, two things happen. First a gap of height dz opens between the surface and the face of the cylinder, where previously there had been a fixed magnetization $\mathbf{M}$, but now there is just free space. Second, between L and $L + dz$ there is now a fixed magnetization $\mathbf{M}$ where previously there was none. Moving the cylinder of fixed $\mathbf{M}$ up a distance dz is thus formally equivalent to keeping the cylinder put and instead introducing a cylinder of the *opposite* magnetization $-\mathbf{M}$ between 0 and dz, while also introducing a cylinder of magnetization $+\mathbf{M}$ between L and $L + dz$. The increase in field energy in this latter case is fairly easily calculated by taking the integral of $\mu_0 \mathbf{M} \dot{c}$

H_0 over the regions in which the fixed magnetization changes.

So the first task is to find the field that exists when the cylinder is pressed flat against the surface. Since there are no free currents in the problem, we may derive $\mathbf{H}$ from a scalar potential satisfying the Laplace equation. To begin we write down the general solutions of the Laplace equation in cylindrical coordinates, observing first that by symmetry we can only keep terms with no azimuthal angle dependence:

$$\Phi(m) = \begin{cases} \int_0^\infty dk\, A(k) e^{-kz} J_0(k\rho), & z > L \\ \int_0^\infty dk\, [B(k)e^{kz} + C(k)e^{-kz}] J_0(k\rho), & 0 < z < L \\ \int_0^\infty dk\, D(k) e^{+kz} J_0(k\rho), & z < 0. \end{cases} \tag{8}$$

The boundary conditions at $z = 0$ are that H_ρ and B_z be continuous. Assuming first of all that the medium existing in the region below $z = 0$ has finite permeability μ, the tangential boundary condition is

$$\left.\frac{\partial \Phi_m}{\partial \rho}\right|_{z=0-} = \left.\frac{\partial \Phi_m}{\partial \rho}\right|_{z=0+}$$

$$\int_0^\infty dk\, k\, D(k) J_1(k\rho) = \int_0^\infty dk\, k\, [B(k) + C(k)] J_1(k\rho). \tag{9}$$

Multiplying (9) by $\rho J_1(k'\rho)$, integrating from 0 to ∞, and using the identity (2), we find

$$D(k) = B(k) + C(k). \tag{10}$$

The normal boundary condition at $z = 0$ is of a mixed type. Below the line we have simply $B_z = \mu H_z$. Above the line we may write $B_z = \mu_0[H_z + M(\rho)]$, where $M(\rho)$ represents the fixed magnetic polarization of the cylinder:

$$M(\rho) = \begin{cases} M, & \rho < a \\ 0, & \rho > a. \end{cases} \tag{11}$$

The normal boundary condition at $z = 0$ is then

$$-\mu \left.\frac{\partial}{\partial z}\Phi_m\right|_{z=0-} = -\mu_0 \left.\frac{\partial}{\partial z}\Phi_m\right|_{z=0+} + \mu_0 M(\rho)$$

$$-\frac{\mu}{\mu_0}\int_0^\infty dk\, k\, D(k) J_0(k\rho) = -\int_0^\infty dk\, k\, [B(k) - C(k)] J_0(k\rho) + M(\rho)$$

Now multiplying by $\rho J_0(k'\rho)$, integrating from $\rho = 0$ to ∞, and using (2) yields

$$\frac{\mu}{\mu_0} D(k) = -B(k) + C(k) - \int_0^\infty \rho\, M(\rho) J_0(k\rho)\, d\rho. \tag{12}$$

Using (11), the integral on the RHS is

$$M \int_0^a \rho J_0(k\rho)\, d\rho = \frac{Ma}{k} J_1(ka) \equiv \gamma(k)$$

where we defined a convenient shorthand. Then (12) is

$$\frac{\mu}{\mu_0} D(k) = -B(k) + C(k) - \gamma(k).$$

Now taking $\mu \to \infty$, we see that, to keep the B and C coefficients from blowing up, we must have $D \to 0$. Then equation (**??**) tells us that $B(k) = -C(k)$, so the middle entry in (8) may be rewritten:

$$\Phi_m(z,\rho) = \int_0^\infty dk\,\beta(k)\sinh(kz)J_0(k\rho), \qquad (0 < z < L).$$

The boundary conditions at $z = L$ are

$$\begin{aligned}
\left.\frac{\partial \Phi_m}{\partial \rho}\right|_{z=L+} &= \left.\frac{\partial \Phi_m}{\partial \rho}\right|_{z=L-} \\
-\left.\frac{\partial \Phi_m}{\partial z}\right|_{z=L+} &= -\left.\frac{\partial \Phi_m}{\partial z}\right|_{z=L-} + M(\rho)
\end{aligned}$$

with $M(\rho)$ defined as above. Working through the same procedure as above yields the conditions

$$\begin{aligned}
A(k)e^{-kL} &= \beta(k)\sinh(kL) \\
A(k)e^{-kL} &= \beta(k)\cosh(kL) + \gamma(k)
\end{aligned}$$

with $\gamma(k)$ defined as above. The solution is

$$\begin{aligned}
\beta(k) &= -\gamma(k)e^{+kL} \\
A(k) &= \gamma(k)\sinh(kL).
\end{aligned}$$

Plugging these back into (8) and differentiating, we find for the z component of the **H** field

$$H_z(\rho,z) = \begin{cases} Ma\displaystyle\int_0^\infty dk\,e^{-kz}\cosh(kL)J_0(k\rho)J_1(ka), & z > L \\ -Ma\displaystyle\int_0^\infty dk\,e^{-kL}\cosh(kz)J_0(k\rho)J_1(ka), & 0 < z < L. \end{cases} \tag{13}$$

Now that we know the field, we want to find the change in energy density incurred by putting into this field a short cylinder (radius a, height dz) of magnetization $-M\hat{\mathbf{k}}$ between $z = 0$ and $z = dz$, and another cylinder of the same size but with magnetization $+M\hat{\mathbf{k}}$ between $z = L$ and $z = L + dz$. The change in field energy is just the integral of $\mu_0 \mathbf{M}\cdot\mathbf{H}$ over the volume in which the magnetization density has changed:

$$\begin{aligned}
dU &= -2\pi\mu_0 M\int_0^{dz}\int_0^a H_z(z,\rho)\rho\,d\rho\,dz + 2\pi\mu_0 M\int_L^{L+dz}\int_0^a H_z(z,\rho)\rho\,d\rho\,dz \\
&= 2\pi\mu_0 M dz\Big(\int_0^a H_z(L,\rho)\rho\,d\rho - \int_0^a H_z(0,\rho)\rho\,d\rho\Big) \qquad (14)
\end{aligned}$$

where in the last step we assumed that H_z remains essentially constant over a distance dz in the z direction, and may thus be taken out of the integral.

Inserting (13) into (), and exchanging the order of integration, we first do the ρ integral:

$$\int_0^a J_0(k\rho)\rho d\rho = \frac{a}{k} J_1(ka).$$

Then () becomes

Solutions to Problems in Jackson, *Classical Electrodynamics*, Third Edition

Homer Reid

March 28, 2002

Chapter 6: Problems 1-8

Problem 6.2

The charge and current densities for a single point charge q can be written formally as

$$\rho(\mathbf{x}', t') = q\delta[\mathbf{x}' - \mathbf{r}(t')]; \qquad \mathbf{J}(\mathbf{x}', t') = q\mathbf{v}(t')\delta[\mathbf{x}' - \mathbf{r}(t')]$$

where $\mathbf{r}(t')$ is the charge's position at time t' and $\mathbf{v}(t')$ is its velocity. In evaluating expressions involving the retarded time, one must put $t' = t_{\text{ret}} = t - R(t')/c$, where $\mathbf{R} = \mathbf{x} - \mathbf{r}(t')$.

(a) As a preliminary to deriving the Heaviside-Feynman expressions for the electric and magnetic fields of a point charge, show that

$$\int d^3x'\, \delta[\mathbf{x}' - \mathbf{r}(t_{\text{ret}})] = \frac{1}{\kappa}$$

where $\kappa = 1 - \mathbf{v} \cdot \hat{\mathbf{R}}/c$. Note that κ is evaluated at the retarded time.

(b) Starting with the Jefimenko generalizations of the Coulomb and Biot-Savart laws, use the expressions for the charge and current densities for a point charge and the result of part a to obtain the Heaviside-Feynman expressions for the electric and magnetic fields of a point charge,

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0}\left\{\left[\frac{\hat{\mathbf{R}}}{\kappa R^2}\right]_{\text{ret}} + \frac{1}{c}\frac{\partial}{\partial t}\left[\frac{\hat{\mathbf{R}}}{\kappa R}\right]_{\text{ret}} - \frac{1}{c^2}\frac{\partial}{\partial t}\left[\frac{\mathbf{v}}{\kappa R}\right]_{\text{ret}}\right\}$$

and

$$\mathbf{B} = \frac{\mu_0 q}{4\pi}\left\{\left[\frac{\mathbf{v} \times \hat{\mathbf{R}}}{\kappa R^2}\right]_{\text{ret}} + \frac{1}{c}\frac{\partial}{\partial t}\left[\frac{\mathbf{v} \times \hat{\mathbf{R}}}{\kappa R}\right]_{\text{ret}}\right\}$$

(c) In our notation Feynman's expression for the electric field is

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0}\left\{\left[\frac{\hat{\mathbf{R}}}{R^2}\right]_{\text{ret}} + \frac{[R]_{\text{ret}}}{c}\frac{\partial}{\partial t}\left[\frac{\hat{\mathbf{R}}}{R^2}\right]_{\text{ret}} + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}[\hat{\mathbf{R}}]_{\text{ret}}\right\}$$

while Heaviside's expression for the magnetic field is

$$\mathbf{B} = \frac{\mu_0 q}{4\pi}\left\{\left[\frac{\mathbf{v} \times \hat{\mathbf{R}}}{\kappa^2 R^2}\right]_{\text{ret}} + \frac{1}{c[R]_{\text{ret}}}\frac{\partial}{\partial t}\left[\frac{\mathbf{v} \times \hat{\mathbf{R}}}{\kappa}\right]_{\text{ret}}\right\}.$$

Show the equivalence of the two sets of expressions for the fields.

(a) Let's first assume that the charge is traveling along the z axis, so that its position is given by

$$\mathbf{r}(t) = (z_0 + v_z t)\hat{\mathbf{k}}.$$

The retarded time $t_{\text{ret}}(t,z)$ at a given point z on the z axis is

$$t_{\text{ret}}(t,z) = t - \frac{z}{c}$$

so

$$\mathbf{r}[t_{\text{ret}}(t,z)] = (z_0 + v_z t_{\text{ret}}(t,z))\hat{\mathbf{k}}.$$

Hence

$$\begin{aligned}
\delta(\mathbf{x} - \mathbf{r}[t_{\text{ret}}(\mathbf{x},t)]) &= \delta(x)\delta(y)\delta\left\{z - [z_0 + v_z t_{\text{ret}}(t,z)]\right\} \\
&= \delta(x)\delta(y)\delta\left\{z - [z_0 + v_z(t - \frac{z}{c})]\right\} \\
&= \delta(x)\delta(y)\delta\left\{z - z_0 - v_z t + \frac{v_z}{c}z)]\right\} \\
&= \delta(x)\delta(y)\delta\left\{\left(1 + \frac{v_z}{c}\right)z - (z_0 + v_z t)\right\}
\end{aligned}$$

By the properties of the δ function we may write this as

$$= \left(\frac{1}{1 + v_z/c}\right)\delta(x)\delta(y)\delta\left\{z - \frac{z_0 + v_z t}{1 + v_z/c}\right\}.$$

The δ function is singling out the point in space from which originates the electromagnetic disturbance we feel at the origin at time t. Let's think about what's going on here in two limiting cases. First, as $v_z \to 0$, the z delta function becomes $\delta(z-(z_0+v_z t))$. This means that the source point for the field we feel at the origin at time t is just $z = z_0 - v_z t$, which is of course just the instantaneous location of the source particle at time t. In other words, the electromagnetic disturbance left behind by the particle at time t reaches the origin so quickly that the particle hasn't had time to move on. The electromagnetic disturbance seems to be coming from the instantaneous location of the particle itself.

In the opposite limit $v_z \to c$, the z delta function becomes $\delta(z-(z_0-v_z t)/2)$. This says that the point from which we feel an electromagnetic disturbance at time t is *half* as far from the origin as the particle itself is at time t. This again makes sense. At each point in the particle's motion, the electromagnetic disturbance it causes begins propagating toward the origin, while meanwhile the particle continues propagating *away* from the origin at the same speed. Hence when the electromagnetic disturbance has reached the origin, the particle has traveled as far as the electromagnetic disturbance did, but in the *opposite* direction, so it is now twice as far from the origin as it was when the disturbance we are just now feeling was generated.

Problem 6.5

A localized electric charge distribution produces an electrostatic field, $\mathbf{E} = -\nabla\Phi$. Into this field is placed a small localized time-independent current density $\mathbf{J}(\mathbf{x})$, which generates a magnetic field $\mathbf{H}$.

(a) Show that the momentum of these electromagnetic fields, (6.117), can be transformed to

$$\mathbf{P}_{\text{field}} = \frac{1}{c^2}\int \Phi\mathbf{J}\, d^3x$$

provided the product $\Phi\mathbf{H}$ falls of rapidly enough at large distances. How rapidly is "rapidly enough"?

(b) Assuming that the current distribution is localized to a region small compared to the scale of variation of the electric field, expand the electrostatic potential in a Taylor series and show that

$$\mathbf{P}_{\text{field}} = \frac{1}{c^2}\mathbf{E}(0) \times \mathbf{m},$$

where $\mathbf{E}(0)$ is the electric field at the current distribution and $\mathbf{m}$ is the magnetic moment, (5.54), caused by the current.

(c) Suupose the current distribution is placed instead in a *uniform* electric field $\mathbf{E}_0$ (filling all space). Show that, no matter how complicated is the localized $\mathbf{J}$, the result in part a is augmented by a surface integral contribution from infinity equal to minus one-third of the result of part b, yielding

$$P_{\text{field}} = \frac{2}{3c^2}\mathbf{E}_0 \times \mathbf{m}.$$

Compare this result with that obtained by working directly with (6.117) and the considerations at the end of Section 5.6.

(a) From the definition of electromagnetic field momentum we have

$$\begin{aligned} c^2\mathbf{P}_{\text{field}} &= \int \mathbf{E} \times \mathbf{H}\, dV \\ &= -\int (\nabla\Phi) \times \mathbf{H}\, dV. \end{aligned}$$

Focusing for now on the z component, we have

$$c^2 P_z = -\int \left(\frac{\partial\Phi}{\partial x}H_y - \frac{\partial\Phi}{\partial y}H_x\right) dx\, dy\, dz \tag{1}$$

Let's take our volume of integration to be a cube of side L, which we will eventually take to infinity. Integrating the first term by parts with respect to x, we have

$$\int_{-L}^{L}\int_{-L}^{L}\left\{\int_{-L}^{L}\frac{\partial\Phi}{\partial x}H_y\,dx\right\}dy\,dz = \int_{-L}^{L}\int_{-L}^{L}\left\{\Phi H_y\Big|_{x=-L}^{x=L} - \int\Phi\frac{\partial H_y}{\partial x}\,dx\right\}dy\,dz.$$

Similarly integrating the second term in (1) by parts with respect to y, we may write (1) as

$$\begin{aligned}c^2P_z &= -\int_{-L}^{L}\int_{-L}^{L}\Phi H_y\Big|_{x=-L}^{x=L}dy\,dz + \int_{-L}^{L}\int_{-L}^{L}\Phi H_x\Big|_{y=-L}^{y=L}dx\,dz + \int\Phi(\nabla\times\mathbf{H})_z\,dV\\ &= -\int_{-L}^{L}\int_{-L}^{L}\Phi H_y\Big|_{x=-L}^{x=L}dy\,dz + \int_{-L}^{L}\int_{-L}^{L}\Phi H_x\Big|_{y=-L}^{y=L}dx\,dz + \int\Phi J_z\,dV\end{aligned}$$

where in going to the last line we used $\nabla\times\mathbf{H}=\mathbf{J}$ since there is no time-dependent $\mathbf{E}$ field. This equation is just the z component of

$$c^2\mathbf{P} = \int\Phi\mathbf{H}\times\,d\mathbf{A} + \int\Phi\mathbf{J}\,dV. \tag{2}$$

If we now take $L\to\infty$, the first integral (which describes surface effects) vanishes providing the product $\Phi(\mathbf{x})\mathbf{H}(\mathbf{x})$ vanishes more quickly (i.e. like a higher power of $\mathbf{x}$) than $\mathbf{x}^2$. Then we are left with just the second term:

$$c^2\mathbf{P} = \int\Phi\mathbf{J}\,dV. \tag{3}$$

(b) We have

$$\Phi(\mathbf{x}) = \Phi(0) + \mathbf{x}\cdot\nabla\Phi(0) + \frac{1}{2}\sum x_i x_j\frac{\partial^2\Phi}{\partial x_i\partial x_j} + \cdots$$

We may arbitrarily choose $\Phi(0)=0$. Also, we are told that the electric field doesn't vary much in the region of nonvanishing $\mathbf{J}$, in which case we may ignore the second derivatives of Φ, to obtain

$$\Phi(\mathbf{x}) \approx \mathbf{x}\cdot\nabla\Phi(0) = -\mathbf{x}\cdot\mathbf{E}(0).$$

Plugging into (3),

$$c^2\mathbf{P} = -\int\big[\mathbf{x}\cdot\mathbf{E}(0)\big]\mathbf{J}\,dV. \tag{4}$$

We have

$$\begin{aligned}-\big[\mathbf{x}\cdot\mathbf{E}(0)\big]\mathbf{J} &= \big[\mathbf{E}(0)\times\mathbf{J}\big]\mathbf{x} - \big[\mathbf{x}\cdot\mathbf{E}(0)\big]\mathbf{J} - \big[\mathbf{E}(0)\times\mathbf{J}\big]\mathbf{x}\\ &= \mathbf{E}(0)\times\big[\mathbf{x}\times\mathbf{J}\big] - \big[\mathbf{E}(0)\times\mathbf{J}\big]\mathbf{x}\end{aligned}$$

where in the first line we added and subtracted a term, and in the second used the BAC-CAB identity of vector analysis. With this, (4) becomes

$$
\begin{aligned}
c^2\mathbf{P} &= \mathbf{E}(0) \times \int \big[\mathbf{x} + \mathbf{J}\big]\, dV - \int \big[\mathbf{E}(0) \times \mathbf{J}\big]\, x\, dV \\
&= 2\mathbf{E}(0) \times \mathbf{m} - \int \big[\mathbf{E}(0) \times \mathbf{J}\big]\, x\, dV
\end{aligned}
$$

where in the first term we have identified the definition of the dipole moment $\mathbf{m}$. Evidently to get this to match up with what Jackson has we need to argue that second term is exactly half the first, but I can't see how to do this for arbitrary $\mathbf{J}$. Can anybody help?

(c) From (2) we have

$$
c^2\mathbf{P} = \int \Phi\mathbf{H} \times \, d\mathbf{A} + \int \Phi\mathbf{J}\, dV.
$$

The second term is just equal to $(\mathbf{E} \times \mathbf{m})/c^2$, as computed in part b. For the first term,

Problem 6.13

A parallel plate capacitor is formed of two flat rectangular perfectly conducting sheets of dimensions a and b separated by a distance d small compared to a or b. Current is fed in and taken out uniformly along adjacent edges of length b. With the input current and voltage defined at this end of the capacitor, calculate the input impedance or admittance using the field concepts of Section 6.9.

(a) Calculate the electric and magnetic fields in the capacitor correct to second order in powers of the frequency, but neglecting fringing fields.

(b) Show that the expansion of the reactance (6.140) in powers of the frequency to an appropriate order is the same as that obtained for a lumped circuit consisting of a capacitance $C = \epsilon_0 ab/d$ in series with an inductance $L = \mu_0 ad/3b$.

(a) We'll suppose the plates are oriented parallel to the xy plane, with the lower plate at $z = 0$ and the upper plate at $z = d$. We'll take the edges of side a parallel to the x axis, and the edges of side b parallel to the y axis. Then the boundary condition on the current density is

$$
\mathbf{J}(0, y, 0) = -\mathbf{J}(0, y, d) = J_0\hat{\mathbf{j}}
$$

for $0 < y < b$.

With neglect of fringing fields, the electric field between the plates exists only in the z direction, while the magnetic field exists only in the x direction. We assume harmonic time dependence and write

$$\mathbf{E}(y) = E(y)e^{-i\omega t}\hat{\mathbf{k}} \qquad \mathbf{B}(y) = B(y)e^{-i\omega t}\hat{\mathbf{x}}; \tag{5}$$

then time differentiation becomes multiplication by $-i\omega$. The Maxwell equations are then

$$\begin{aligned}
\nabla \cdot \mathbf{E} = 0 \qquad & \Rightarrow \qquad \frac{\partial E}{\partial z} = 0\\
\nabla \cdot \mathbf{B} = 0 \qquad & \Rightarrow \qquad \frac{\partial B}{\partial x} = 0\\
\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad & \Rightarrow \qquad \frac{\partial E}{\partial y} = +i\omega B\\
\nabla \times \mathbf{B} = \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} \qquad & \Rightarrow \qquad \frac{\partial B}{\partial y} = +\frac{i\omega}{c^2}E.
\end{aligned} \tag{6}$$

We postulate an expansion in powers of ω for E and B:

$$\begin{aligned}
E(y) &= E_0(y) + \omega E_1(y) + \omega^2 E_2(y) + \cdots\\
B(y) &= B_0(y) + \omega B_1(y) + \omega^2 B_2(y) + \cdots
\end{aligned} \tag{7}$$

Then the curl equations in (6) become

$$\begin{aligned}
\frac{\partial}{\partial y}\big[E_0 + \omega E_1 + \omega^2 E_2 + \cdots\big] &= i\omega\big[B_0 + \omega B_1 + \omega^2 B_2 + \cdots\big]\\
\frac{\partial}{\partial y}\big[B_0 + \omega B_1 + \omega^2 B_2 + \cdots\big] &= \frac{i\omega}{c^2}\big[E_0 + \omega E_1 + \omega^2 E_2 + \cdots\big]
\end{aligned}$$

Equating equal powers of ω in these equations, we obtain

$$
\begin{aligned}
\frac{\partial E_0}{\partial y} &= 0 & &\Rightarrow & E_0 &= \alpha \\
\frac{\partial B_0}{\partial y} &= 0 & &\Rightarrow & B_0 &= \beta \\
\frac{\partial E_1}{\partial y} &= iB_0 = i\beta & &\Rightarrow & E_1 &= i\beta y \\
\frac{\partial B_1}{\partial y} &= \frac{i}{c^2}E_0 = \frac{i\alpha}{c^2} & &\Rightarrow & B_1 &= \frac{i\alpha}{c^2}y \\
\frac{\partial E_2}{\partial y} &= iB_1 = -\frac{\alpha}{c^2}y & &\Rightarrow & E_2 &= -\frac{\alpha}{2c^2}y^2 \\
\frac{\partial B_2}{\partial y} &= \frac{i}{c^2}E_1 = -\frac{\beta}{c^2}y & &\Rightarrow & B_2 &= -\frac{\beta}{2c^2}y^2 \\
\frac{\partial E_3}{\partial y} &= iB_2 = -\frac{i\beta}{2c^2}y^2 & &\Rightarrow & E_3 &= -\frac{i\beta}{6c^2}y^3 \\
\frac{\partial B_3}{\partial y} &= \frac{i}{c^2}E_2 = -\frac{i\alpha}{2c^4}y^2 & &\Rightarrow & B_3 &= -\frac{i\alpha}{6c^4}y^3 \\
\frac{\partial E_4}{\partial y} &= iB_3 = \frac{\alpha}{24c^4}y^3 & &\Rightarrow & E_4 &= \frac{i\beta}{24c^2}y^4 \\
\frac{\partial B_4}{\partial y} &= \frac{i}{c^2}E_3 = +\frac{\beta}{6c^4}y^3 & &\Rightarrow & B_4 &= \frac{\beta}{24c^4}y^4
\end{aligned}
$$

and so on. Plugging into (7), we obtain

$$
\begin{aligned}
E(y) &= \alpha(1 - \frac{(ky)^2}{2} + \frac{(ky)^4}{24} + \cdots) + i\beta c(ky - \frac{(ky)^3}{6} + \cdots) \qquad (8) \\
&= \alpha\cos ky + i\beta c \sin ky \qquad (9) \\
B(y) &= \beta \cos ky + \frac{i\alpha}{c}\sin ky \qquad (10)
\end{aligned}
$$

where $k = \omega/c$, and where we simply wrote down what we guessed to be the sums of the full infinite series from their first few terms.

To complete the problem we need to determine the constants α and β, for which purpose we appeal to the boundary conditions on the plates. We know that the discontinuities in the **E** and **B** field are proportional to the surface charge and current densities on the plates. Since these conditions only give information on the *differences* between the fields outside and between the plates, we ostensibly have to know what the fields are outside to get what they are inside. But for the purposes of this problem we'll just assume there are no fields outside, so the charge and current densities on the plates determine the fields inside. I know this is correct in the low-frequency limit, and in the high-frequency limit I'm not yet sure how to compute the radiation fields in the region outside the plates, so I will ignore them.

The boundary conditions are

$$E_z = -\frac{\sigma}{\epsilon_0}$$
$$B_x = -\mu_0 K_y$$

where σ and K_y are the charge density and y component of the surface current density on the top plate (assumed to be identical but with opposite sign on the bottom plate). Plugging in the solutions (9) and (**??**), we have

$$\begin{aligned} \sigma &= -\epsilon_0(\alpha \cos ky + i\beta c \sin ky) \\ K_y &= -\frac{1}{\mu_0}(\beta \cos ky + \frac{i\alpha}{c}\sin ky) \end{aligned} \tag{11}$$

As a sanity check, we can verify the continuity relation between charge and current on the plates:

$$\frac{\partial K_y}{\partial y} = -\frac{\partial \sigma}{\partial t} = +i\omega\sigma$$

Plugging in (11), the left and right sides of this are

$$\text{LHS} = -\frac{1}{\mu_0}(-k\beta \sin ky + \frac{ik\alpha}{c}\cos ky)$$
$$\text{RHS} = -i\omega\epsilon_0(\alpha \cos ky + i\beta c \sin ky)$$

and the two are evidently equal.

The forcing function in this problem is the surface current density specified on the edges of length b. If the total current fed into the $y = 0$ edge of the top plate is $I(t) = I_0 \cos \omega t$ (with an opposite current taken out of the $y = 0$ edge of the bottom plate) then the surface current boundary conditions are

$$K_y(y = 0) = \frac{I_0}{b}\cos \omega t$$
$$K_y(y = a) = 0$$

Comparing with (11), we see that these boundary conditions we have to take

$$\beta = -\frac{\mu_0 I_0}{b}\cos \omega t$$
$$\alpha = -\frac{i\mu_0 I_0 c}{b}\cos \omega t \cot ka$$

Plugging into (9) and (10),

$$\begin{aligned}
E_z &= -\frac{i\mu_0 I_0 c}{b}\cos\omega t\,[\cot ka\cos ky + \sin ky] \\
&= -\frac{i\mu_0 I_0 c}{b}\cos\omega t\left[\frac{1}{\sin ka}\right][\cos ka\cos ky + \sin ka\sin ky] \\
&= -\frac{i\mu_0 I_0 c}{b}\cos\omega t\frac{\cos[k(y-a)]}{\sin ka} \\
B_z &= -\frac{\mu_0 I_0}{b}\cos\omega t\,[-\cos ky + \cot ka\sin ky] \\
&= -\frac{\mu_0 I_0}{b}\cos\omega t\frac{1}{\sin ka}[-\sin ka\cos ky + \cos ka\sin ky] \\
&= -\frac{\mu_0 I_0}{b}\cos\omega t\frac{\sin[k(y-a)]}{\sin ka}
\end{aligned}$$

Problem 6.14

An ideal circular parallel plate capacitor of radius a and plate separation $d \ll a$ is connected to a current source by axial leads, as shown in the sketch. The current in the wire is $I(t) = I_0 \cos\omega t$.

(a) Calculate the electric and magnetic fields between the plates to second order in powers of the frequency (or wave number), neglecting the effects of fringing fields.

(b) Calculate the volume integrals of w_e and w_m that enter the definition of the reactance X, (6.140), to second order in ω. Show that in terms of the input current I_i, defined by $I_i = -i\omega Q$, where Q is the *total charge* on one plate, these energies are

$$\int w_e\, d^3x = \frac{1}{4\pi\epsilon_0}\frac{|I_i|^2 d}{\omega^2 a^2}, \qquad \int w_m\, d^3x = \frac{\mu_0}{4\pi}\frac{|I_i|^2 d}{8}\left(1 + \frac{\omega^2 a^2}{12c^2}\right)$$

(c) Show that the equivalent series circuit has $C \approx \pi\epsilon_0 a^2/d$, $L \approx \mu_0 d/8\pi$, and that an estimate for the resonant frequency of the system is $\omega_{\text{res}} = 2\sqrt{2}c/a$. Compare with the first root of $J_0(x)$.

(a) We work in cylindrical coordinates and assume harmonic time dependence ($\propto e^{-i\omega t}$) for all quantities; then time differentiation is replaced by multiplication by $-i\omega$. If we neglect the effects of fringing fields, everything is symmetric in θ, and the electric field between the plates is entirely in the z direction, while the magnetic field is entirely in the θ direction:

$$\mathbf{E}(\mathbf{x},t) = E(r,z)e^{-i\omega t}\hat{\mathbf{z}} \qquad \mathbf{B} = B(r,z)e^{-i\omega t}\hat{\theta}. \tag{12}$$

The Maxwell equations for the fields between the plates are

$$\begin{aligned}
\nabla\cdot\mathbf{E} = 0 \quad &\Rightarrow \quad \frac{\partial}{\partial z}E = 0 \\
\nabla\cdot\mathbf{B} = 0 \quad &\Rightarrow \quad \frac{\partial}{\partial \theta}B = 0 \\
\nabla\times\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad &\Rightarrow \quad \frac{\partial E}{\partial r} = -i\omega B \\
\nabla\times\mathbf{B} = \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} \quad &\Rightarrow \quad \frac{1}{r}\frac{\partial}{\partial r}(rB) = -\frac{i\omega}{c^2}E.
\end{aligned} \tag{13}$$

To proceed, let's propose an expansion of the fields in powers of the frequency:

$$E(r) = E_0(r,z) + \omega E_1(r,z) + \omega^2 E_2(r,z) + \cdots \tag{14}$$

$$B(r) = B_0(r,z) + \omega B_1(r,z) + \omega^2 B_2(r,z) + \cdots \tag{15}$$

Then the curl equations in (13) become

$$\frac{\partial}{\partial r}\big[E_0 + \omega E_1 + \omega^2 E_2\big] = -i\omega\big[B_0 + \omega B_1 + \omega^2 B_2\big]$$

$$\frac{1}{r}\frac{\partial}{\partial r}\big[rB_0 + \omega r B_1 + \omega^2 r B_2\big] = -\frac{i\omega}{c^2}\big[E_0 + \omega E_1 + \omega^2 E_2\big]$$

Now we just have to go through and equate like powers of ω in these equations.

For $n=0$, we have

$$\frac{\partial E_0}{\partial r} = 0 \qquad \Rightarrow \qquad E_0 = \alpha_1 \tag{16}$$

for some constant α_1, and

$$\frac{1}{r}\frac{\partial}{\partial r}(rB_0) = 0 \qquad \Rightarrow \qquad B_0 = \frac{\beta}{r}. \tag{17}$$

But for nonzero β this blows up at the origin. Hence we must take $\beta = 0$, so $B_0 = 0$. 2 For $n = 1$, we have

$$\frac{\partial E_1}{\partial r} = -iB_0 = 0 \qquad \Rightarrow \qquad E_1 = \alpha_2 \tag{18}$$

for some constant α_2, and

$$\frac{1}{r}\frac{\partial}{\partial r}(rB_1) = -\frac{i}{c^2}E_0 = -\frac{i\alpha_1}{c^2} \qquad \Rightarrow \qquad B_1 = -\frac{i\alpha_1}{2c^2}r. \tag{19}$$

Continuing,

$$\frac{\partial E_2}{\partial r} = -iB_1 = -\frac{\alpha_1}{2c^2}r \qquad \Rightarrow \qquad E_2 = -\frac{\alpha_1}{4c^2}r^2 \tag{20}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(rB_2) = -\frac{i}{c^2}E_1 = -\frac{i\alpha_2}{c^2} \qquad \Rightarrow \qquad B_2 = -\frac{i\alpha_2}{2c^2}r \tag{21}$$

$$\frac{\partial E_3}{\partial r} = -iB_2 = -\frac{\alpha_2}{2c^2}r \qquad \Rightarrow \qquad E_3 = -\frac{\alpha_2}{4c^2}r^2 \tag{22}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(rB_3) = -\frac{i}{c^2}E_2 = \frac{i\alpha_1}{4c^4}r^2 \qquad \Rightarrow \qquad B_3 = \frac{i\alpha_1}{16c^4}r^3 \tag{23}$$

Evidently E_{2n} and E_{2n+1} have the same form but differ by the presence of α_1 or α_2, as is true for B_{2n-1} and B_{2n}. Plugging in equations (16) through (23) into (14) and (15), we obtain

$$\begin{aligned} E(r) &= (\alpha_1 + \omega\alpha_2)\big(1 - \frac{\omega^2}{4c^2}r^2 + \frac{\omega^4}{64c^4}r^4\big) + \cdots \\ &= (\alpha_1 + \omega\alpha_2)\big(1 - \frac{(kr)^2}{4} + \frac{(kr)^4}{64} + \cdots\big) \\ B(r) &= -\frac{i}{c}(\alpha_1 + \omega\alpha_2)\left(\frac{kr}{2}\right)\big(1 - \frac{(kr)^2}{8} + \cdots\big) \end{aligned}$$

These look the first few terms in certain Bessel functions:

$$\begin{aligned} E(r) &= (\alpha_1 + \omega\alpha_2)J_0(kr) \equiv \beta J_0(kr) \\ B(r) &= -\frac{i}{c}\beta J_1(kr) \end{aligned}$$

where we can define the constant $\beta = (\alpha_1 + \omega\alpha_2)$ since we're dealing with a fixed frequency. Inserting into (12) we obtain

$$\mathbf{E}(r,t) = \beta J_0(kr)e^{-i\omega t}\hat{\mathbf{k}} \qquad \mathbf{B}(r,t) = -\frac{i}{c}\beta J_1(kr)e^{-i\omega t}\hat{\theta}. \tag{24}$$

To work out the value of β, we need to apply the boundary conditions at the capacitor plates. An easy way to do this is to consider what happens as $\omega \to 0$. In that limit there is no magnetic field, and the electric field between the plates is just $E_z(t) = -2\sigma(t)/\epsilon_0$, where $\sigma(t)$ is the instantaneous value of the surface charge induced on each plate (positive on the top plate, negative on the bottom). Now, the total charge on the top plate is just the integral of the current flowing onto that plate:

$$q = \int I(t)\,dt = \frac{I_0}{\omega}\sin\omega t$$

and the surface charge is this divided by the plate area (since we are assuming a low frequency, any charge that flows onto the plate instantaneously equilibrates with the rest of the charge on the plate, yielding a constant surface charge density):

$$\sigma(t) = \frac{I_0}{\pi a^2\omega}\sin\omega t.$$

Hence the electric field in the low frequency limit is

$$E_z(\omega \to 0) = -\frac{2I_0}{\pi a^2\omega\epsilon_0}\sin\omega.$$

Comparing this with(24) in the limit $k \to 0$, we obtain

$$\beta = -\frac{2I_0 i}{\pi a^2 \omega \epsilon_0}.$$

Hence

$$\mathbf{E}(r,t) = -\frac{2I_0}{\pi a^2 \omega \epsilon_0} J_0(kr) \sin \omega t\, \hat{\mathbf{k}} \qquad \mathbf{B}(r,t) = \frac{2\mu_0 I_0 c}{\pi a^2 \omega} J_1(kr) \cos \omega t\, \hat{\theta}. \quad (25)$$

(b) The average energy densities are

$$w_e = \frac{\epsilon_0}{4} E^2 = \frac{I_0^2}{(\pi a^2 \omega)^2 \epsilon_0} \Big(1 - \frac{kr}{4} + \cdots\Big)^2$$

$$w_m = \frac{1}{4\mu_0^2} B^2 = \frac{\mu_0 I_0^2 c^2}{(\pi a^2 \omega)^2} \left(\frac{kr}{2}\right)^2 \Big(1 - \frac{(kr)^2}{8} + \cdots\Big)^2$$

We only have to keep the first terms in the parentheses to get the energy right to second order in ω:

$$U_e \approx \frac{I_0^2}{(\pi a^2 \omega)^2 \epsilon_0} \int_0^a (2\pi d)(r\, dr)$$

$$= \frac{I_0^2 d}{\pi a^2 \omega^2 \epsilon_0}$$

$$U_m = \frac{\mu_0 I_0^2 c^2}{(\pi a^2 \omega)^2} \int_0^a (2\pi d)(rdr)\Big(\frac{kr}{2} - \frac{(kr)^3}{8} + \cdots\Big) U_m \quad = \frac{\mu_0 I_0^2 c^2}{(\pi a^2 \omega)^2} \int_0^a (2\pi d)(rdr)\Big(\frac{kr}{2} - \frac{(kr)^3}{8} + \cdots\Big)$$

Solutions to Problems in Jackson, *Classical Electrodynamics*, Third Edition

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May 24, 2002

Chapter 8: Waveguide Derivations

Before starting the problems, I thought it would be useful to run through my own derivations of some of the formulas from this chapter.

Waveguides and cavities: basic pedagogy

The unifying feature of waveguide and cavity problems is that we single out one spatial coordinate and announce from the start that the fields will have sinusoidal dependence on that coordinate. Taking the special coordinate to be z, this means that all components of all fields have the functional form $f(x,y)e^{ikz}$ for some wavevector k.

Assuming harmonic time dependence, we write explicitly

$$\begin{aligned} \mathbf{E}(\mathbf{x}) &= \Big\{ E_x(x,y)\mathbf{i} + E_y(x,y)\mathbf{j} + E_z(x,y)\mathbf{k} \Big\} e^{i(kz-\omega t)} \\ \mathbf{B}(\mathbf{x}) &= \Big\{ B_x(x,y)\mathbf{i} + B_y(x,y)\mathbf{j}) + B_z(x,y)\mathbf{k} \Big\} e^{i(kz-\omega t)} \end{aligned} \tag{1}$$

We have here a total of six functions $f(x,y)$ that we must find to satisfy Maxwell's equations with the relevant boundary conditions. At first this would appear tough since the six fields are all coupled by Maxwell's equations, but after a little algebra we obtain the following simplified situation: The $z-$ direction fields $E_z(x,y)$ and $B_z(x,y)$ turn out to satisfy (separately) simple one-dimensional differential equations, which may be readily solved upon specifying the boundary conditions for a particular situation. Meanwhile, the remaining fields (E_x, E_y, B_x, B_y) can be expressed simply as linear combinations of E_z and B_z and their derivatives, so once we obtain the z fields we have everything. In what follows we'll derive the differential equations satisfied by E_z and B_z and the equations giving the remaining fields in terms of them.

The differential equations for E_z and B_z

The Maxwell curl equations are

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \qquad \nabla \times \mathbf{B} = \frac{1}{c_m^2}\frac{\partial \mathbf{E}}{\partial t}$$

where c_m is the speed of light in the medium. We begin by applying the first curl equation to our ansatz (1), obtaining

$$\partial_y E_z - ikE_y = i\omega B_x \tag{2}$$
$$-\partial_x E_z + ikE_x = i\omega B_y \tag{3}$$
$$\partial_x E_y - \partial_y E_x = i\omega B_z, \tag{4}$$

and we pause to solve the first two of these for E_x and E_y :

$$E_x = \frac{\omega}{k}B_y - \frac{i}{k}\partial_x E_z, \qquad E_y = -\frac{\omega}{k}B_x - \frac{i}{k}\partial_y E_z. \tag{5}$$

Next we apply the second curl equation to our ansatz, obtaining

$$\partial_y B_z - ikB_y = -\frac{i\omega}{c_m^2}E_x \tag{6}$$
$$-\partial_x B_z + ikB_x = -i\frac{\omega}{c_m^2}E_y \tag{7}$$
$$\partial_x B_y - \partial_y B_x = -i\frac{\omega}{c_m^2}E_z. \tag{8}$$

But in (5) we solved for E_x and E_y, and if we then plug those solutions into (6) and (7) we can solve for B_x and B_y in terms of B_z and E_z :

$$B_x = \frac{ikc_m^2}{\omega^2 - k^2c_m^2}\left(\partial_x B_z + \frac{\omega}{c_m^2 k}\partial_y E_z\right) \tag{9}$$
$$B_y = \frac{ikc_m^2}{\omega^2 - k^2c_m^2}\left(\partial_y B_z - \frac{\omega}{c_m^2 k}\partial_x E_z\right). \tag{10}$$

Finally, with the ansatz (1) the equation $\nabla \cdot \mathbf{B} = 0$ reads

$$\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} = -ikB_z.$$

When we plug (9) and (10) into this, the terms involving E_z fields cancel, and we obtain an equation involving B_z alone:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)B_z + \left(\frac{\omega^2}{c_m^2} - k^2\right)B_z = 0$$

or

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)B_z + \gamma^2 B_z = 0 \tag{11}$$

where

$$\gamma = \sqrt{\frac{\omega^2}{c^2} - k^2}.$$

If we had carried out this derivation in the reverse order we would have obtained the same equation for E_z :

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) E_z + \gamma E_z = 0. \tag{12}$$

We can think of equations (11) and (12) as eigenvalue equations that have solutions only for certain values of the parameter γ, which depend on the boundary conditions.

Armed with equations (11) and (12) and the boundary conditions appropriate to our problem we can now solve for B_z and E_z and then use (9) and (10) to find the remaining components of the **B** field. The remaining components of the **E** field are given by analogous equations:

$$E_x = \frac{ikc_m^2}{\omega^2 - k^2 c_m^2}\left(\partial_x E_z + \frac{\omega}{k}\partial_y B_z\right) \tag{13}$$

$$E_y = \frac{ikc_m^2}{\omega^2 - k^2 c_m^2}\left(\partial_y E_z - \frac{\omega}{k}\partial_x B_z\right). \tag{14}$$

Boundary Conditions; TE, TM, TEM Modes

The boundary conditions on the fields at the surfaces of the waveguide or cavity are that $\mathbf{E}_\parallel$ and $\mathbf{B}_\perp$ be continuous, where $\perp$ denotes the component of the vector normal to the boundary surface and $\parallel$ includes all other components of the vector. This means that the two eigenvalue equations (11) and (12) must be solved subject to different boundary conditions, which means in general their eigenvalues will be different. If we have a solution of (12) for some value of γ (i.e. for some combination of values of ω and k), then there will be no nonzero solution of (11) for that value of γ, and hence we must have $B_z = 0$ identically for the mode at that frequency and wavevector. Since in this case the magnetic field has nonzero components only transverse to the direction of propagation, this is called a transverse magnetic mode. Similarly, if (11) can be solved with nonzero B_z at some γ, then $E_z = 0$ and we have a transverse electric mode. A mode for which *both* E_z and B_z are zero is called a transverse electromagnetic mode, and can only exist in the region *between* two conducting surfaces, not within a single conductor as is possible for TE and TM modes.

Since either E_z or B_z is zero, we can simplify some of the equations above and collect results appropriate to the two cases.

TM Modes	**TE Modes**
$B_z \equiv 0$	$E_z \equiv 0$
$\nabla_t^2 E_z + \gamma^2 E_z = 0, \quad E_n \Big\vert_{\partial S} = 0$	$\nabla_t^2 B_z + \gamma^2 B_z = 0, \quad \frac{\partial B_n}{\partial n}\Big\vert_{\partial S} = 0$
$E_x = \frac{ikc_m^2}{\omega^2 - k^2 c_m^2}\partial_x E_z$	$E_x = \frac{i\omega c_m^2}{\omega^2 - k^2 c_m^2}\partial_y B_z$
$E_y = \frac{ikc_m^2}{\omega^2 - k^2 c_m^2}\partial_y E_z$	$E_y = -\frac{i\omega c_m^2}{\omega^2 - k^2 c_m^2}\partial_x B_z$
$B_x = \frac{i\omega}{\omega^2 - k^2 c_m^2}\partial_y E_z$	$B_x = \frac{ikc_m^2}{\omega^2 - k^2 c_m^2}\partial_x B_z$
$B_y = -\frac{i\omega}{\omega^2 - k^2 c_m^2}\partial_x E_z$	$B_y = \frac{ikc_m^2}{\omega^2 - k^2 c_m^2}\partial_y B_z$

(A factor of $e^{i(kz-\omega t)}$ is understood in all of these expressions.)

For TM modes, the boundary condition is $\mathbf{E}_{\parallel} = 0$, and E_z is always perpendicular to the boundary surfaces, so the boundary condition for the eigenvalue equation is $E_z = 0$. For the case of TE modes, the boundary condition is $\mathbf{B}_{\perp} = 0$. Suppose one boundary surface is the yz plane. The normal to this plane is the x direction, so B_x must vanish at this surface; but we just saw that in the TE case $B_x \propto \partial_x B_z$, i.e. the derivative of B_z normal to the boundary surface must vanish. This is general: the boundary condition for the eigenvalue equation in the TM case is $\partial B_z/\partial n = 0$.

Power flow; Energy Loss

The flow of power down a waveguide is described by the z component of the Poynting vector $\mathbf{S} = \mathbf{E} \times \mathbf{H} = \frac{1}{\mu}\mathbf{E} \times \mathbf{B}$. Using the boxed expressions above, for the two types of modes we obtain

$$
\begin{aligned}
S_z^{\rm TM} &= \frac{1}{\mu}(E_x B_y - E_y B_x) \\
&= \frac{k\omega c_m^2}{\mu(\omega^2 - k^2 c_m^2)^2}\left[(\partial_x E_z)^2 + (\partial_y E_z)^2\right] e^{2i(kz-\omega t)}
\end{aligned}
$$

or, in the time average,

$$
\begin{aligned}
&= \frac{k\omega c_m^2}{2\mu(\omega^2 - k^2 c_m^2)^2}\left[(\partial_x E_z)^2 + (\partial_y E_z)^2\right] \\
&= \frac{\epsilon k\omega}{2\gamma^4}\left[(\partial_x E_z)^2 + (\partial_y E_z)^2\right]. \qquad (15)
\end{aligned}
$$

Similarly, for TE modes we obtain

$$S_z^{\text{TE}} = \frac{k\omega}{2\mu\gamma^4}\left[(\partial_x B_z)^2 + (\partial_y B_z)^2\right]. \tag{16}$$

To address the issue of dissipation in the boundaries, we solve Maxwell's equations within the boundary surfaces. The two curl equations are

$$\nabla \times \mathbf{E} = i\omega\mathbf{B} \tag{17}$$

$$\begin{aligned}\nabla \times \mathbf{B} &= \mu\mathbf{J} - i\mu\epsilon\omega\mathbf{E} \\ &= \mu\left(\sigma - i\epsilon\omega\right)\mathbf{E} \\ &\approx \mu\sigma\mathbf{E}\end{aligned} \tag{18}$$

since $\sigma \gg \epsilon\omega$ in most cases. (For example, for a copper waveguide with air or vacuum interior we have we have $\sigma \approx 6 \cdot 10^7\ \Omega^{-1}\ \text{m}^{-1}$, while $\epsilon\omega \approx 9 \cdot 10^{-12}\ \Omega^{-1}\ \text{m}^{-1} \cdot (\omega$ in rad/sec), so the approximation is good up to frequencies $\omega \sim 10^{19}$ rad/sec.)

Now we assume that the fields are only changing significantly in the direction normal to the boundary surface (i.e., as we go deeper and deeper into the boundary surface the fields die out rapidly, whereas as we move along parallel to the boundary surface the fields don't change much) and keep only the normal derivative in the curl equations. If ρ measures the depth of penetration into the surface, the curl equations become

$$\begin{aligned}\hat{\rho} \times \frac{\partial\mathbf{E}}{\partial\rho} &= i\omega\mathbf{B} \\ \hat{\rho} \times \frac{\partial\mathbf{B}}{\partial\rho} &= \mu\sigma\mathbf{E}\end{aligned}$$

Differentiating the first of these, taking the cross product with $\hat{\rho}$ of both sides, and substituting in the second equation yields

$$\hat{\rho} \times \hat{\rho} \times \frac{\partial^2\mathbf{E}}{\partial\rho^2} = i\omega\mu\sigma\mathbf{E}$$

or, using the bac-cab rule,

$$\frac{\partial^2 E_\rho}{\partial\rho^2}\hat{\rho} - \frac{\partial^2\mathbf{E}}{\partial\rho^2} = i\omega\mu\sigma\mathbf{E}.$$

Evidently the ρ component of the LHS vanishes here, so $E_\rho = 0$; the electric field within the conducting boundary has no component normal to the surface. For the remaining components we obtain

$$\frac{\partial^2\mathbf{E}_\parallel}{\partial\rho^2} + i\omega\mu\sigma\mathbf{E}_\parallel = 0$$

with solution

$$\begin{aligned}\mathbf{E}_{\parallel} &= e^{\pm\sqrt{i\omega\mu\sigma}\rho}\mathbf{E}_0 \\ &= e^{\pm(1+i)\frac{\rho}{\delta}}\mathbf{E}_0\end{aligned}$$

where $\delta = \sqrt{2/\omega\mu\sigma}$ is the skin depth and $\mathbf{E}_0$ is the field just at the surface of the boundary. To keep the solution from blowing up as we penetrate into the conductor we take the negative sign in the exponent. From (17) we then obtain

$$\mathbf{B} = \frac{i-1}{\delta\omega}(\hat{\rho}\times\mathbf{E}_0)e^{-(1+i)\frac{\rho}{\delta}}.$$

Evaluating this at the surface yields the modified boundary condition on the fields in the cavity or waveguide:

$$\mathbf{B}_0 = \frac{i-1}{\delta\omega}(\hat{\rho}\times\mathbf{E}_0). \tag{19}$$

From this equation we can work out the power loss per unit length in the cavity or waveguide. The power dissipated in a volume dV is $\int(\mathbf{J}\cdot\mathbf{E})\,dV = \sigma\int E^2\,dV$. We integrate over the volume occupied by the boundary surfaces in a length dz :

$$\begin{aligned}dP &= dz\left\{\oint\int_0^\infty \sigma E_0^2 e^{-2(1+i)\frac{\rho}{\delta}}\,d\rho\,dl\right\}e^{2i(kz-\omega t)} \\ &= dz\left\{\frac{\sigma\delta}{2(1+i)}\oint E_0^2\,dl\right\}e^{2i(kz-\omega t)}\end{aligned}$$

or, taking the time average,

$$\frac{dP}{dz} = \frac{\sigma\delta}{4\sqrt{2}}\oint E_0^2\,dl \tag{20}$$

where the line integral is over the cross section of the surface boundary at a fixed value of z.

Solutions to Problems in Jackson,
Classical Electrodynamics, Third Edition

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May 24, 2002

Chapter 8: Problems 1-8

Problem 8.2

A transmission line consisting of two concentric cylinders of metal with conductivity σ and skin depth δ, as shown, is filled with a uniform lossless dielectric (μ, ϵ). A TEM mode is propagated along this line.

(a) Show that the time-averaged power flow along the line is

$$P = \sqrt{\frac{\mu}{\epsilon}}\pi a^2 |H_0|^2 \ln\left(\frac{b}{a}\right)$$

where H_0 is the peak value of the azimuthal magnetic field at the surface of the inner conductor.

(b) Show that the transmitted power is attenuated along the line as

$$P(z) = P_0 e^{-2\gamma z}$$

where

$$\gamma = \frac{1}{2\sigma\delta}\sqrt{\frac{\epsilon}{\mu}}\frac{\left(\frac{1}{a}+\frac{1}{b}\right)}{\ln\left(\frac{b}{a}\right)}.$$

(c) The characteristic impedance Z_0 of the line is defined as the ratio of the voltage between the cylinders to the axial current flowing in one of them at any position z. Show that for this line

$$Z_0 = \frac{1}{2\pi}\sqrt{\frac{\mu}{\epsilon}}\ln\left(\frac{b}{a}\right).$$

(d) Show that the series resistance and inductance per unit length of the line are

$$R = \frac{1}{2\pi\sigma\delta}\left(\frac{1}{a}+\frac{1}{b}\right)$$

$$L = \left\{\frac{\mu}{2\pi}\ln\left(\frac{b}{a}\right) + \frac{\mu_c\delta}{4\pi}\left(\frac{1}{a}+\frac{1}{b}\right)\right\}.$$

(a) For the TEM mode, the electric field in the waveguide may be written

$$\mathbf{E}(x,y,z,t) = \mathbf{E}_t(x,y)e^{-ikz}e^{-i\omega t}$$

where $\mathbf{E}_t$ has only x and y components and may be derived from a scalar potential, i.e. $\mathbf{E}_t = -\nabla_t\Phi$. Since Φ satisfies the Laplace equation, we may write its general form immediately (neglecting an arbitrary constant):

$$\Phi(\rho,\theta) = \beta_0 \ln\rho + \sum_{l=1}^{\infty}(\alpha_l\rho^l + \beta_l\rho^{-l})\sin(l\theta + \alpha_l).$$

In this part of the problem we'll neglect dissipation in the waveguide walls, so the boundary condition on $\mathbf{E}_t$ is that its components transverse to the walls vanish, i.e.

$$\left.\frac{\partial\Phi}{\partial\theta}\right|_{r=b} = \left.\frac{\partial\Phi}{\partial\theta}\right|_{r=a} = 0.$$

This yields no condition on β_0, since the θ derivative of that term vanishes anyway, but on the terms in the summation we obtain the conditions

$$\alpha_l a^l + \beta_l a^{-l} = \alpha_l b^l + \beta_l b^{-l} = 0$$

which can only be satisfied if $\alpha_l = \beta_l = 0$ for $l \neq 0$. Hence we have

$$\Phi(\rho) = \beta_0 \ln\rho \qquad \longrightarrow \qquad \mathbf{E} = -\beta_0 \frac{1}{\rho}\hat{\rho}. \tag{1}$$

The magnetic field is found from Jackson's (8.28):

$$\begin{aligned}\mathbf{H} = -\frac{1}{\mu}\mathbf{B} &= \sqrt{\frac{\epsilon}{\mu}}(\mathbf{z}\times\mathbf{E}) \\ &= \sqrt{\frac{\epsilon}{\mu}}\frac{\beta_0}{\rho}\hat{\theta}.\end{aligned} \tag{2}$$

The time-averaged Poynting vector is

$$\mathbf{S} = \frac{1}{2}(\mathbf{E}\times\mathbf{H}^*) = \frac{1}{2}\sqrt{\frac{\epsilon}{\mu}}|\beta_0|^2\left(\frac{1}{\rho}\right)^2\hat{\mathbf{z}}$$

Integrating over the cross section of the waveguide, we obtain the power transfer:

$$\begin{aligned}P &= \int_a^b\int_0^{2\pi}\mathbf{S}\cdot d\mathbf{A} \\ &= \frac{1}{2}\sqrt{\frac{\epsilon}{\mu}}|\beta_0|^2\int_a^b\frac{2\pi\rho\,d\rho}{\rho^2} \\ &= \sqrt{\frac{\epsilon}{\mu}}|\beta_0|^2\cdot\pi\ln\left(\frac{b}{a}\right) \\ &= \sqrt{\frac{\mu}{\epsilon}}(\pi a^2)\ln\left(\frac{b}{a}\right)\cdot\left[\frac{\epsilon}{\mu}\frac{|\beta_0|^2}{a^2}\right]\end{aligned} \tag{3}$$

Referring back to (2) to rewrite the term in brackets, we obtain

$$P = \sqrt{\frac{\mu}{\epsilon}}(\pi a^2)\ln\left(\frac{b}{a}\right)|H(a)|^2 \tag{4}$$

(b) Without going back and completely re-solving for the fields in the waveguide for the case of finite conductivity, we can calculate the power loss per unit length approximately using Jackson's equation (8.58):

$$\begin{aligned}
-\frac{dP}{dz} &= \frac{1}{2\sigma\delta}\oint_c |\hat{\rho}\times\mathbf{H}|^2\,dl \\
&= \frac{1}{2\sigma\delta}\left(\frac{\epsilon}{\mu}\right)\left(2\pi b\cdot\left(\frac{\beta_0}{b}\right)^2 + 2\pi a\cdot\left(\frac{\beta_0}{a}\right)^2\right) \\
&= \frac{\pi\beta_0^2}{\sigma\delta}\left(\frac{\epsilon}{\mu}\right)\left(\frac{1}{b}+\frac{1}{a}\right).
\end{aligned}$$

Dividing by (3), we obtain

$$\gamma = -\frac{1}{2P}\frac{dP}{dz} = \frac{1}{2\sigma\delta}\sqrt{\frac{\epsilon}{\mu}}\frac{\left(\frac{1}{a}+\frac{1}{b}\right)}{\ln\left(\frac{b}{a}\right)}.$$

(c) The fields inside the waveguide are

$$\begin{aligned}
\mathbf{E}(\rho,z,t) &= -\frac{\beta_0}{\rho}e^{i(kz-\omega t)}\hat{\rho} \\
\mathbf{H}(\rho,z,t) &= \sqrt{\frac{\epsilon}{\mu}}\frac{\beta_0}{\rho}e^{i(kz-\omega t)}\hat{\theta}
\end{aligned}$$

From the **E** field we can compute the voltage difference between the cylinders:

$$V(z,t) = -\beta_0 e^{i(kz-\omega t)}\int_a^b \frac{d\rho}{\rho} = -\beta_0 e^{i(kz-\omega t)}\ln\frac{b}{a} \tag{5}$$

while from the **H** field we can compute the axial current flowing in, say, the outer cylinder:

$$I = 2\pi b|\mathbf{K}_b| = 2\pi b|\hat{\rho}\times\mathbf{H}(\rho=b)| = 2\pi\sqrt{\frac{\epsilon}{\mu}}\beta_0 e^{i(kz-\omega t)}. \tag{6}$$

Dividing (5) by (6), we have

$$Z = \frac{V}{I} = \frac{1}{2\pi}\sqrt{\frac{\mu}{\epsilon}}\ln\frac{b}{a}.$$

Problem 8.4

Transverse electric and magnetic waves are propagated along a hollow, right circular cylinder of brass with inner radius R.

(a) Find the cutoff frequencies of the various TE and TM modes. Determine numerically the lowest cutoff frequency (the dominant mode) in terms of the tube radius and the ratio of cutoff frequencies of the next four higher modes to that of the dominant mode. For this part assume that hte conductivity of brass is infinite.

(b) Calculate the attenuation constant of the waveguide as a function of frequency for the lowest two modes and plot it as a function of frequency.

(a) The equation we have to solve is

$$(\nabla_t^2 + \gamma^2)\Psi(\rho, \theta) = 0,$$

i.e. the Helmholtz equation. Ψ is E_z for the TM case and H_z for the TE case. The boundary conditions are $\Psi(\rho = R) = 0$ for the TM case, and $(\partial\Psi/\partial\rho)(\rho = R) = 0$ for the TE case.

The general solution of Helmholtz in 2D is

$$\Psi(\rho, \theta) = \sum_{L=0}^{\infty} J_L(\gamma\rho)(A_L e^{iL\theta} + B_L e^{-iL\theta}) + N_L(\gamma\rho)(C_L e^{iL\theta} + D_L e^{iL\theta}).$$

Since this solution must be valid everywhere in the interior of the waveguide, including at $\rho = 0$, the part of the solution involving N_L must vanish. Also, for a physical solution we must have L an integer. But otherwise I don't think there are any constraints on A_L and B_L. I guess these guys are determined by the field configuration one forces into the waveguide at one of its ends.

The allowed values of γ are determined by the boundary conditions. These are

$$\text{TM case}: \qquad \Psi|_{\rho=R} = 0 \qquad \Longrightarrow \qquad J_L(\gamma R) = 0 \tag{7}$$

$$\text{TE case}: \qquad \left.\frac{\partial\Psi}{\partial\rho}\right|_{\rho=R} = 0 \qquad \Longrightarrow \qquad J_L'(\gamma R) = 0 \tag{8}$$

$$\tag{9}$$

Hence the allowable eigenvalues are given by

$$\gamma_i = \frac{x_i}{R}$$

where the x_i are the roots of $J_L(x) = 0$ and $J'_L(x) = 0$. Referring to Jackson's tables on pages 114 and 370, we can write down the five lowest-lying eigenvalues:

$$\begin{aligned}
\gamma_1 &= \frac{1.841}{R}, &\quad& \text{TE, } L = 1\\
\gamma_2 &= \frac{2.405}{R}, && \text{TM, } L = 0\\
\gamma_3 &= \frac{3.054}{R}, && \text{TE, } L = 2\\
\gamma_{4a} &= \frac{3.832}{R}, && \text{TE, } L = 1\\
\gamma_{4b} &= \frac{3.832}{R}, && \text{TM, } L = 0.
\end{aligned}$$

The last two eigenvalues are degenerate.

The lowest cutoff frequency is

$$\omega_c = \frac{\gamma_1}{\sqrt{\mu\epsilon}} = \frac{1.841}{R\sqrt{\mu\epsilon}}.$$

(b) The lowest-lying mode is the TE mode with $L = 1$. For this mode we have

$$H_z(\rho, \theta, z, t) = H_0 J_1(\gamma_1 \rho) e^{i\theta} e^{i(kz - \omega t)} \tag{10}$$

with $k^2 = \mu\epsilon\omega^2 - \gamma_1^2$. The tangential component of the field, from Jackson (8.33), is

$$H_\theta(\rho, \theta, z, t) = -\frac{k}{\rho\gamma^2} H_z \tag{11}$$

Using (10) and (11), we can find the current induced in the wall of the conductor at $\rho = R$:

$$\mathbf{K}_{\text{eff}} = \hat{\rho} \times \mathbf{H}(\rho = R) = -H_0 J_1(\gamma_1 R) e^{i\theta} e^{i(kz - \omega_t)} \left[\hat{\theta} + \frac{k}{R\gamma_1^2}\hat{\mathbf{z}}\right].$$

Then Jackson (8.58) is

$$-\frac{dP}{dz} = \frac{1}{2\sigma\delta} H_0^2 J_1^2(\gamma_1 R) \cdot 2\pi R \left[1 + \left(\frac{k}{R\gamma_1^2}\right)^2\right] \tag{12}$$

On the other hand, the transmitted power is given by Jackson (8.51):

$$\begin{aligned}
P &= \frac{1}{2}\sqrt{\frac{\mu}{\epsilon}} \left(\frac{\omega}{\omega_\lambda}\right)^2 \left[1 - \left(\frac{\omega_\lambda}{\omega}\right)^2\right]^{1/2} \int_A H_z^* H_z \, dA\\
&= \pi H_0^2 \sqrt{\frac{\mu}{\epsilon}} \left(\frac{\omega}{\omega_\lambda}\right)^2 \left[1 - \left(\frac{\omega_\lambda}{\omega}\right)^2\right]^{1/2} \int_0^R \rho J_1^2(\gamma_1 \rho) \, d\rho
\end{aligned}$$

The integral can be evaluated with Jackson (3.95):

$$= \pi H_0^2 \sqrt{\frac{\mu}{\epsilon}} \left(\frac{\omega}{\omega_\lambda}\right)^2 \left[1 - \left(\frac{\omega_\lambda}{\omega}\right)^2\right]^{1/2} \left\{\frac{R^2}{2}[J_2(\gamma_1 R)]^2\right\} \tag{13}$$

Dividing (12) by (13), we calculate the attenuation coefficient:

$$\begin{aligned}
\beta = \frac{1}{2P}\frac{dP}{dz} &= \frac{2}{\sigma\delta R}\sqrt{\frac{\epsilon}{\mu}}\left[\frac{J_1(1.841)}{J_2(1.841)}\right]^2 \left[1 + \left(\frac{k}{R\gamma_1^2}\right)^2\right] \left(\frac{\omega_\lambda}{\omega}\right)^2 \left[\frac{\omega^2}{\omega^2 - \omega_\lambda^2}\right]^{1/2} \\
= \frac{1}{2P}\frac{dP}{dz} &= \frac{2}{\sigma\delta R}\sqrt{\frac{\epsilon}{\mu}}\left[\frac{J_1(1.841)}{J_2(1.841)}\right]^2 \left[1 + \left(\frac{Rk}{(1.841)^2}\right)^2\right] \left(\frac{\omega_\lambda}{\omega}\right)^2 \left[\frac{\omega^2}{\omega^2 - \omega_\lambda^2}\right]^{1/2} \\
= \frac{1}{2P}\frac{dP}{dz} &= \frac{2}{\sigma\delta R}\sqrt{\frac{\epsilon}{\mu}}\left[\frac{J_1(1.841)}{J_2(1.841)}\right]^2 \left[\frac{\mu\epsilon R^2\omega^2}{(1.841)^2}\right] \left(\frac{\omega_\lambda}{\omega}\right)^2 \left[\frac{\omega^2}{\omega^2 - \omega_\lambda^2}\right]^{1/2}.
\end{aligned}$$

Problem 8.5

A waveguide is constructed so that the cross section of the guide forms a right triangle with sides of length a, a, $\sqrt{2}a$, as shown. The medium inside has $\mu_r = \epsilon_r = 1$.

(a) Assuming infinite conductivity for the walls, determine the possible modes of propagation and their cutoff frequencies.

(b) For the lowest mode of each type calculate the attenuation constant, assuming that the walls have large, but finite, conductivity. Compare the result with that for a square guide of side a made from the same material.

(a) We'll take the origin of coordinates at the lower left corner of the triangle. Then the boundary surfaces are defined by $x = 0$, $y = a$, and $x = y$. The task is to solve $(\nabla_t^2 + \gamma^2)\Psi = 0$ subject to the vanishing of Ψ, or its normal derivative, at the walls. In the text, Jackson finds the form of the solutions for a rectangular waveguide. A little bit of staring at the triangular waveguide reveals that appropriate solutions for this geometry can be assembled from linear combinations of the solutions for the rectangular case. For example, a term like $\sin k_x x \sin k_y y$, for suitable choices of k_x and k_y, already vanishes on the two legs of the triangle. To get it to vanish on the third boundary surface, i.e. the hypotenuse ($x = y$), we can simply subtract the same term with k_x and k_y swapped. In other words, we take

$$E_z(x,y) = \sum A_{mn}\left[\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{a}\right) - \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{a}\right)\right] \qquad \text{(TM)}$$

$$H_z(x,y) = \sum B_{mn}\left[\cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{a}\right) + \cos\left(\frac{n\pi x}{a}\right)\cos\left(\frac{m\pi y}{a}\right)\right] \qquad \text{(TE)}$$

The TE case involves the plus sign because in the normal derivative on the diagonal boundary surface the x derivative comes in with the opposite sign as the y derivative.

These satisfy $(\nabla_t^2 + \gamma_{mn}^2)\Psi = 0$, where

$$\gamma_{mn}^2 = \left(\frac{\pi}{a}\right)^2 (n_x^2 + n_y^2).$$

In contrast to the rectangular case, TM modes with $m = n$ vanish identically. For both TM and TE modes, mode (m, n) is the same mode as (n, m).

As in the case of the rectangular waveguide, the smallest value of γ is to be had for the $\text{TE}_1, 0$ mode, in which case

$$\gamma_{10} = \left(\frac{\pi}{a}\right)$$

and the cutoff frequency is $\omega_{c(1,0)} = \pi/(a\sqrt{\mu\epsilon})$. For the TM case the lowest propagating mode is $(m, n) = (2, 1)$, for which $\gamma_{21} = \sqrt{5}\pi/a$ and $\omega_{c(2,1)} = \sqrt{5}\omega_{c(1,0)}$.

(b) The lowest-frequency TE mode has

$$H_z = H_0\left[\cos\left(\frac{\pi x}{a}\right) + \cos\left(\frac{\pi y}{a}\right)\right]$$

$$\mathbf{H}_t = \frac{ik\pi}{a\gamma_{10}^2} H_0\left[\sin\left(\frac{\pi x}{a}\right)\hat{\mathbf{i}} + \sin\left(\frac{\pi y}{a}\right)\hat{\mathbf{j}}\right].$$

The power loss is

$$-\frac{dP}{dz} = \frac{1}{2\sigma\delta}\oint |\mathbf{n}\times\mathbf{H}|^2\, dl \tag{14}$$

On the lower surface $(y = 0)$ we have

$$|\mathbf{n}\times\mathbf{H}|^2 = |H_y^2 + H_z^2| = H_0^2\left\{\left[\cos\left(\frac{\pi x}{a}\right) + 1\right]^2 + \frac{k^2\pi^2}{a^2\gamma_{10}^4}\sin^2\left(\frac{\pi x}{a}\right)\right\}$$

The contribution of the lower surface to the integral in (14) is thus

$$\int_{\text{lower}} = aH_0^2\left[\frac{3}{2} + \frac{k^2\pi^2}{a^2\gamma_{10}^4}\right] \tag{15}$$

The contribution of the right (vertical) boundary surface is the same. On the diagonal boundary surface, we have

$$\mathbf{n} = \frac{1}{\sqrt{2}}(-\hat{\mathbf{i}} + \hat{\mathbf{j}}) \qquad \Longrightarrow \qquad \mathbf{n}\times\mathbf{H} = \frac{1}{\sqrt{2}}[H_z\hat{\mathbf{i}} + H_z\hat{\mathbf{j}} - (H_y + H_x)\hat{\mathbf{k}}]$$

with magnitude

$$\begin{aligned} |\mathbf{n}\times\mathbf{H}|^2 &= \frac{1}{2}[H_x^2 + H_y^2 + H_z^2 + 2H_xH_y] \\ &= \frac{H_0^2}{2}\left\{4\cos^2\left(\frac{\pi\gamma}{a}\right) + 4\left(\frac{k\pi}{a\gamma_{10}}\right)^2\sin^2\left(\frac{\pi\gamma}{a}\right)\right\} \end{aligned}$$

where $\gamma = x = y$ is the common coordinate as we move from $(0,0)$ to (a,a). In the integral in (14) we can put $dl = \sqrt{2}d\gamma$ and integrate over γ from 0 to a to obtain

$$\int_{\text{diagonal}} = \sqrt{2}aH_0^2\left[1+\frac{k^2\pi^2}{a^2\gamma_{10}^4}\right].$$

Adding this to two times (15) and inserting into (14), we have

$$-\frac{dP}{dz} = \frac{aH_0^2}{2\sigma\delta}\left[3+\sqrt{2}+(2+\sqrt{2})\frac{k^2\pi^2}{a^2\gamma_{10}^4}\right]. \tag{16}$$

On the other hand, from Jackson (8.51) we have

$$P = \frac{H_0^2}{2}\sqrt{\frac{\mu}{\epsilon}}\left(\frac{\omega}{\omega_\lambda}\right)^2\left(1-\frac{\omega_\lambda^2}{\omega^2}\right)^{1/2}$$
$$\times\int_0^a\int_0^y\left[\cos^2\left(\frac{\pi x}{a}\right)+\cos^2\left(\frac{\pi y}{a}\right)+2\cos\left(\frac{\pi x}{a}\right)\cos\left(\frac{\pi y}{a}\right)\right]dx\,dy$$

By symmetry, the integral is just half of what we would get from integrating the integrand over a square of side a, which, by inspection, is a^2. Hence

$$P = \frac{a^2H_0^2}{4}\sqrt{\frac{\mu}{\epsilon}}\left(\frac{\omega}{\omega_\lambda}\right)^2\left(1-\frac{\omega_\lambda^2}{\omega^2}\right).^{1/2}$$

We could at this point proceed to write out the explicit form of the attenuation constant, but what's the point?

Problem 8.6

A resonant cavity of copper consists of a hollow, right circular cylinder of inner radius R and length L, with flat end faces.

(a) Determine the resonant frequencies of the cavity for all types of waves. With $(1/\sqrt{\mu\epsilon}R)$ as a unit of frequency, plot the lowest four resonant frequencies of each type as a function of R/L for $0 < R/L < 2$. Does the same mode have the lowest frequency for all R/L ?

(b) If R=2 cm, L=3 cm, and the cavity is made of pure copper, what is the numerical value of Q for the lowest resonant mode?

(a) Taking the origin at the center of the cavity, the ρ and ϕ components of the fields must vanish at $z = \pm L/2$. Since the z dependence of all field components is $e^{\pm ikz}$, the allowed values of k are $k = n\pi/L$, with $\mathbf{E} \propto \sin kz$ for k even and $\mathbf{E} \propto \cos kz$ for k odd.

The equation characterizing TM modes is

$$(\nabla_t^2 + \gamma^2)E_z = 0, \qquad E_z|_{\partial S} = 0.$$

Expanding this in cylindrical coordinates, we obtain

$$\frac{\partial^2 E_z}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial E_z}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 E_z}{\partial \phi^2} + \gamma^2 E_z = 0$$

We put $E_z(\rho, \phi) = R(\rho)P(\phi)$ to obtain

$$\begin{aligned} \frac{\partial^2 P}{\partial \phi^2} + \nu P &= 0 \\ \frac{\partial^2 R}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial R}{\partial \rho} + \left(\gamma^2 - \frac{\nu^2}{\rho^2}\right) &= 0. \end{aligned}$$

The solutions are

$$\begin{aligned} P(\phi) &= e^{\pm i\nu\phi} \\ R(\rho) &= J_\nu(\gamma\rho). \end{aligned}$$

For single-valuedness we require $\nu \in \mathbb{Z}$, and to ensure $E_z(\rho = R) = 0$ we require $\gamma = x_{\nu m}/R$ where $x_{\nu m}$ is the mth root of $J_\nu(x) = 0$. Hence

$$E_z = AJ_\nu\left(x_{\nu m}\frac{\rho}{R}\right)e^{\pm i\nu\phi}e^{-i\omega t}\begin{cases}\sin kz, & k\text{even} \\ \cos kz, & k\text{odd}\end{cases} \qquad \text{(TM modes)}.$$

For TE modes the relevant equation is

$$(\nabla_t^2 + \gamma^2)B_z = 0, \qquad \left.\frac{\partial B_z}{\partial n}\right|_{\partial S} = 0.$$

The general solution to the differential equation is the same as above, but now the boundary condition requires $J'_\nu(\gamma R) = 0$, so $\gamma = y_{\nu m}/R$ where $y_{\nu m}$ is the mth root of $J'_\nu(y) = 0$. Then the solutions look like

$$B_z = AJ_\nu\left(y_{\nu m}\frac{\rho}{R}\right)e^{\pm i\nu\phi}e^{-i\omega t}\begin{cases}\sin kz, & k\text{even} \\ \cos kz, & k\text{odd}\end{cases} \qquad \text{(TE modes)}.$$

As we saw above, the allowed wavevectors are $k = n\pi/L$. The frequency is related to the wavenumber according to

$$\begin{aligned} \omega_{\nu m n} &= c_m\sqrt{\gamma_{\nu m}^2 + k_n^2} \\ &= \begin{cases} \dfrac{c_m}{R}\sqrt{x_{\nu m}^2 + \pi^2\left(\dfrac{R}{L}\right)^2 n^2}, & \text{(TM)} \\ \dfrac{c_m}{R}\sqrt{y_{\nu m}^2 + \pi^2\left(\dfrac{R}{L}\right)^2 n^2}, & \text{(TE)} \end{cases} \end{aligned}$$

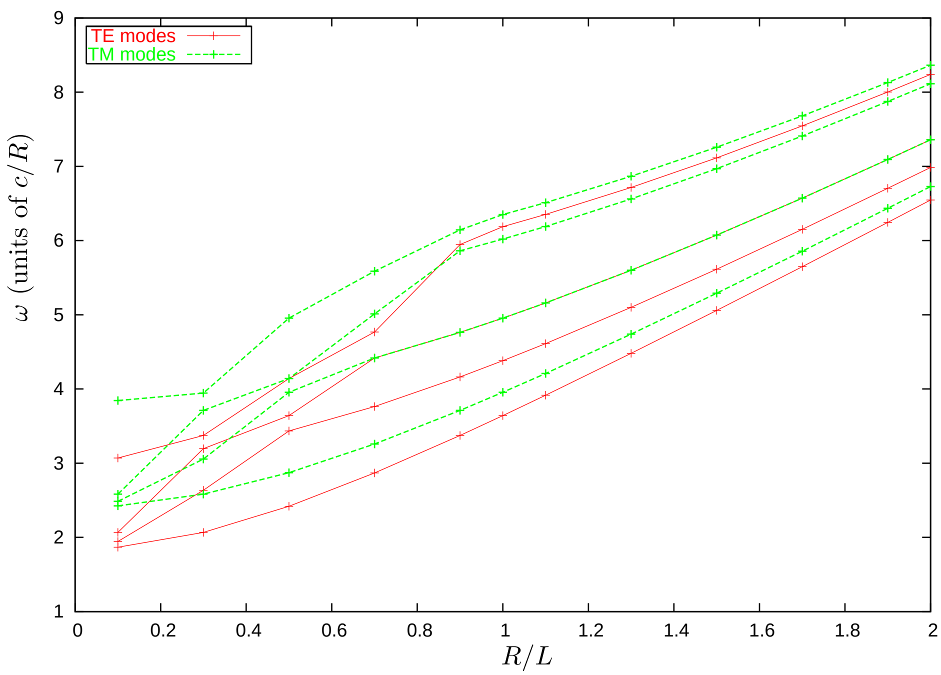

Figure 1: TM and TE mode frequencies for the resonant cavity of Problem 8.6.

(The m subscript on c_m is not related to the m subscripts on ω and ν).

The lowest four TM and TE mode frequencies are shown in Figure 1.

(b) The lowest resonant mode is the $\mathrm{TE}_{1,1,1}$ mode.info

Solutions to Problems in Jackson, *Classical Electrodynamics*, Third Edition

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January 13, 2003

Chapter 11: Problems 1-8

Problem 11.4

A possible clock is shown in the figure. It consists of a flashtube F and a photocell P shielded so that each views only the mirror M, located a distance d away, and mounted rigidly with respect to the flashtube-photocell assembly. The electronic innards of the box are such that when the photocell responds to a light flash from the mirror, the flashtube is triggered with a negligible delay and emits a short flash toward the mirror. The clock thus "ticks" once every $(2d/c)$ seconds when at rest.

(a) Suppose that the clock moves with a uniform velocity v, perpendicular to the line from PF to M, relative to an observer. Using the second postulate of relativity, show by explicit geometrical or algebraic construction that the observer sees the relativistic time dilatation as the clock moves by.

(b) Suppose that the clock moves with a velocity v parallel to the line from PF to M. Verify that here, too, the clock is observed to tick more slowly, by the same time dilatation factor.

(a) Suppose that, relative to an observer, the clock is moving with speed v perpendicular to the direction in which the light travels back and forth. Let dt be the time measured by the observer for the light to travel from the flashtube to the mirror. The vertical distance traveled by the light is d, but—as far as the observer is concerned—during the time dt the mirror has also translated a horizontal distance vdt. Hence the total distance the observer sees the light travel (one-way) is $\sqrt{d^2 + (vdt)^2}$. But this distance must equal cdt, since the light must have the universal speed c in any reference frame. Hence we have

$$cdt = \sqrt{d^2 + (vdt)^2}$$

or

$$dt = \left(\frac{1}{1 + \frac{v^2}{c^2}} \right)^{1/2} \frac{d}{c}.$$

This is the time (as measured by the observer) in which the light makes the trip from flashtube to mirror. The light takes the same amount of time to travel back to the photocell, so the total period of the clock is just twice this, or

$$\text{period} = \left(\frac{1}{1 - \frac{v^2}{c^2}} \right)^{1/2} \frac{2d}{c}$$

which is greater than the period of the clock at rest by the normal Lorentz dilatation factor.

(b) Now we suppose that the clock is moving relative to an observer with speed v *parallel to* the direction of motion of the light. We align the z axis with the direction of motion. Then we may write down the space-time coordinates (in the clock's rest frame) of the three relevant events in the operation of the clock (taking $x^4 = ct$):

$$\begin{aligned} x_a &= (0,0,0,0), && \text{(light leaves flashtube)} \\ x_b &= (0,0,d,d), && \text{(light reaches mirror)} \\ x_c &= (0,0,0,2d), && \text{(light reaches photocell).} \end{aligned}$$

The transformation matrix from the clock's reference frame to the observer's reference frame is

$$\Gamma = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma & \gamma\beta \\ 0 & 0 & \gamma\beta & \gamma\beta \end{pmatrix}$$

and using this we may write down the spacetime coordinates of the three events above in the rest frame of the observer:

$$\begin{aligned} x'_a &= (0,0,0,0) \\ x'_b &= \Big(0,0,\gamma(1+\beta)d, \gamma(1+\beta)d\Big) \\ x'_c &= \Big(0,0,2\gamma\beta d, 2\gamma d\Big) \end{aligned}$$

Evidently the difference in the time coordinates of events a and c (which is just the observed period of the clock) is $c\Delta t = 2\gamma d$, so again the observer observes the clock to have a period of $2\gamma d/c$, longer by the factor γ than the period of the clock in its rest frame.

Problem 11.5

A coordinate system K' moves with a velocity $\mathbf{v}$ relative to another system K. In K' a particle has a velocity $\mathbf{u}'$ and an acceleration $\mathbf{a}'$. Find the Lorentz transformation law for acceleration, and show that in the system K the components of acceleration parallel and perpendicular to $\mathbf{v}$ are

$$\mathbf{a}_{\|} = \frac{\left(1 - \frac{v^2}{c^2}\right)^{3/2}}{\left(1 + \frac{\mathbf{v}\cdot\mathbf{u}'}{c^2}\right)^3}\mathbf{a}'_{\|}$$

$$\mathbf{a}_{\perp} = \frac{\left(1 - \frac{v^2}{c^2}\right)}{\left(1 + \frac{\mathbf{v}\cdot\mathbf{u}'}{c^2}\right)^3}\left(\mathbf{a}'_{\perp} + \frac{\mathbf{v}}{c^2} \times (\mathbf{a}' \times \mathbf{u}')\right).$$

The initial components of the velocity in the moving frame (frame K') are $u'_{\|}$ and $\mathbf{u}'_{\perp}$. Using Jackson's equations 11.31 to transform these to frame K, we obtain

$$u_{\|}(0) = \frac{u'_{\|} + v}{1 + \frac{u'_{\|} v}{c^2}} \tag{1}$$

$$\mathbf{u}_{\perp}(0) = \frac{\mathbf{u}'_{\perp}}{\gamma_v\left(1 + \frac{u_{\|} v}{c^2}\right)}. \tag{2}$$

After a time dt has passed in frame K, a time $dt' = dt/\gamma$ has passed in frame $\mathbf{K}'$, and the velocity has increased by the amount $\mathbf{a}'dt' = \mathbf{a}'dt/\gamma$. Then we can write down the new components of the velocity in K' and again transform to K:

$$u_{\|}(dt) = \frac{u'_{\|} + a'_{\|}dt' + v}{1 + \frac{(u'_{\|} + a'_{\|}dt')v}{c^2}} \tag{3}$$

$$\mathbf{u}_{\perp}(dt) = \frac{\mathbf{u}'_{\perp}}{\gamma_v\left(1 + \frac{(u'_{\|} + a'_{\|}dt')v}{c^2}\right)} \tag{4}$$

Subtracting (1) from (3), we have

$$\Delta u_\| = \frac{u'_\| + a'_\| dt' + v}{1 + \frac{(u'_\| + a'_\| dt')v}{c^2}} - \frac{u'_\| + v}{1 + \frac{u'_\| v}{c^2}}$$
$$= \frac{a'_\| dt' \left(1 - \frac{v^2}{c^2}\right)}{\left(1 + \frac{u'_\| v}{c^2}\right)^2 + \left(1 + \frac{u'_\| v}{c^2}\right) a'_\| dt'}$$

Using the relation $dt' = dt/\gamma$ we can rewrite this as

$$= \frac{a'_\| dt \left(1 - \frac{v^2}{c^2}\right)^{3/2}}{\left(1 + \frac{u'_\| v}{c^2}\right)^2 + \left(1 + \frac{u'_\| v}{c^2}\right)\left(1 + \frac{v^2}{c^2}\right)^{1/2} a'_\| dt}$$

Dividing by dt and taking the limit as $dt \to 0$, we obtain

$$a_\| = \lim_{dt \to 0} \frac{\Delta u_\|}{dt} = \frac{a'_\| \left(1 - \frac{v^2}{c^2}\right)^{3/2}}{\left(1 + \frac{u'_\| v}{c^2}\right)^2}.$$

So evidently I'm off by 1 in the exponent of the denominator. What am I doing wrong?

Next, subtracting (2) from (4), we have

$$\Delta \mathbf{u}_\perp = \frac{1}{\gamma_v}\left[\frac{\mathbf{u}'_\perp + \mathbf{a}'_\perp dt'}{1 + \frac{u'_\| v}{c^2} + \frac{a'_\| dt' v}{c^2}} - \frac{\mathbf{u}'_\perp}{1 + \frac{u'_\| v}{c^2}}\right]$$
$$= \frac{1}{\gamma_v}\left[\frac{\mathbf{a}'_\perp dt' \left(1 + \frac{u'_\| v}{c^2}\right) - \mathbf{u}'_\perp \left(\frac{a'_\| dt' v}{c^2}\right)}{\left(1 + \frac{u'_\| v}{c^2}\right)^2 + \left(1 + \frac{u'_\| v}{c^2}\right)\frac{a'_\| dt' v}{c^2}}\right]$$
$$= \frac{1}{\gamma_v}\left[\frac{\mathbf{a}'_\perp dt' + \frac{v}{c^2}\left(u'_\| \mathbf{a}'_\perp - \mathbf{u}'_\perp a'_\|\right) dt'}{\left(1 + \frac{u'_\| v}{c^2}\right)^2 + \left(1 + \frac{u'_\| v}{c^2}\right)\frac{a'_\| dt' v}{c^2}}\right]$$

Again putting in $dt = \gamma dt'$, we have

$$= \frac{1}{\gamma_v^2}\left[\frac{\mathbf{a}'_\perp dt + \frac{v}{c^2}\left(u'_\| \mathbf{a}'_\perp - \mathbf{u}'_\perp a'_\|\right) dt}{\left(1 + \frac{u'_\| v}{c^2}\right)^2 + \left(1 + \frac{u'_\| v}{c^2}\right)\frac{a'_\| dt v}{\gamma_v c^2}}\right]$$

As before, when we divide by dt and take the limit as $dt \to 0$ the second term in the denominator becomes irrelevant and we obtain

$$\begin{aligned}\mathbf{a}_\perp = \lim_{dt\to 0}\frac{\Delta \mathbf{u}_\perp}{dt} &= \left[\frac{1}{\gamma_v^2\left(1+\frac{u'_\parallel v}{c^2}\right)^2}\right]\left[\mathbf{a}'_\perp + \frac{v}{c^2}\left(u'_\parallel \mathbf{a}'_\perp - \mathbf{u}'_\perp a'_\parallel\right)\right] \\ &= \left[\frac{1-\frac{v^2}{c^2}}{\left(1+\frac{u'_\parallel v}{c^2}\right)^2}\right]\left[\mathbf{a}'_\perp + \frac{v}{c^2}\left(u'_\parallel \mathbf{a}'_\perp - \mathbf{u}'_\perp a'_\parallel\right)\right]\end{aligned}$$

Jackson writes the second term in the brackets a little differently. To show that his expression is equivalent to mine, we use the BAC-CAB rule:

$$\begin{aligned}\mathbf{v}\times(\mathbf{a}'\times\mathbf{u}') &= (\mathbf{v}\cdot\mathbf{u}')\mathbf{a}' - (\mathbf{v}\cdot\mathbf{a}')\mathbf{u}' \\ &= vu'_\parallel\left(\mathbf{a}'_\parallel + \mathbf{a}'_\perp\right) - va'_\parallel\left(\mathbf{u}'_\parallel + \mathbf{u}'_\perp\right)\end{aligned}$$

The parallel components cancel, and we are left with

$$= v(u'_\parallel \mathbf{a}'_\perp - a'_\parallel \mathbf{u}'_\perp)$$

which is the way I wrote it above.

But I'm still off by 1 in the exponent of the term in the denominator! What am I missing?

Problem 11.6

Assume that a rocket ship leaves the earth in the year 2010. One of a set of twins born in 2080 remains on earth; the other rides in the rocket. The rocket ship is so constructed that it has an acceleration g in its own rest frame (this makes the occupants feel at home). It accelerates in a straight-line path for 5 years (by its own clocks), decelerates at the same rate for 5 more years, turns around, accelerates for 5 years, decelerates for 5 years, and lands on earth. The twin in the rocket is 40 years old.

(a) What year is it on earth?

(b) How far away from the earth did the rocket ship travel?

Let $v(t)$ be the speed of the rocket, as observed on earth, at a time t as measured on earth. Using the first result of problem 11.5, we see that the acceleration of the rocket as measured from earth is

$$a = \frac{dv}{dt} = \pm\left(1 - \frac{v(t)^2}{c^2}\right)^{3/2} g,$$

or

$$\frac{d\beta}{dt} = \pm \left(1-\beta^2\right)^{3/2}\alpha$$

where $\alpha = g/c$ and the plus or minus sign depends on whether we are accelerating or decelerating. Manipulating this a little, we obtain

$$\frac{d\beta}{(1-\beta^2)^{3/2}} = \pm\alpha dt.$$

Integrating, we obtain

$$\left|\frac{\beta'}{(1-\beta'^2)^{1/2}}\right|_{\beta_1}^{\beta_2} = \pm\alpha(t_2 - t_1). \tag{5}$$

For the first leg of the rocket's journey, we have $t_1 = \beta_1 = 0$ and we take the plus sign in (5). Then we find

$$\frac{\beta}{(1-\beta^2)^{1/2}} = \alpha t$$

or

$$\beta(t) = \frac{\alpha t}{[1+(\alpha t)^2]^{1/2}} \tag{6}$$

and

$$\gamma(t) = \frac{1}{\sqrt{1-\beta^2(t)}} = \sqrt{1+(\alpha t)^2} \tag{7}$$

Next let's work out the relation between time as measured on the rocket and time as measured on earth. With primed (unprimed) quantities referring to the rocket (to earth), the infinitesimal relation is

$$dt = \gamma(t)dt'$$

or

$$\begin{aligned} t_2' - t_1' &= \int_{t_1}^{t_2} \frac{dt}{\gamma(t)} \\ &= \int_{t_1}^{t_2} \frac{dt}{\sqrt{1+(\alpha t)^2}} \\ &= \frac{1}{\alpha}\int_{\alpha t_1}^{\alpha t_2} \frac{du}{\sqrt{1+u^2}} \\ &= \frac{1}{\alpha}\left[\sinh^{-1}(\alpha t_2) - \sinh^{-1}(\alpha t_1)\right] \end{aligned}$$

For the first leg of the journey this becomes

$$t_2 = \frac{1}{\alpha}\sinh(\alpha t_2'). \tag{8}$$

Now we know that the first leg of the journey lasts until $t_2' = 5$ years, and we have

$$\alpha = \frac{g}{c} = \frac{9.8 \text{ m s}^{-2}}{3 \cdot 10^8 \text{ m s}^{-1}} = 3.27 \cdot 10^{-8} \text{ s}^{-1}.$$

Then the time on earth at the end of the first leg of the journey is, from (8),

$$\begin{aligned} t_2 &= \left(\frac{1}{3.27 \cdot 10^{-8}}\right) \cdot \sinh\left[3.27 \cdot 10^{-8} \text{ s}^{-1} \,(5 \text{ yr}) \left(\frac{3.153 \cdot 10^7 \text{ s}}{1 \text{ yr}}\right)\right] \text{ s} \\ &= 2.65 \cdot 10^9 \text{ s} \\ &\approx 84 \text{ yr}. \end{aligned}$$

Finally, the distance the rocket travels during the first leg of its journey is

$$\begin{aligned} d &= c\int_0^t \beta(t) dt \\ &= c\int_0^t \frac{\alpha t}{\sqrt{1+(\alpha t)^2}}\, dt \\ &= \frac{c}{\alpha}\int_0^{\alpha t} \frac{u}{\sqrt{1+u^2}}\, du \\ &= \frac{c}{\alpha}\left\{\left[1+(\alpha t)^2\right]^{1/2} - 1\right\} \\ &= 3.0 \cdot 10^{15} \text{ meters}. \end{aligned}$$

The behavior of the rocket on the subsequent three legs of the journey is similar to that in the first leg. In particular, the total distance traveled away from earth is twice that covered in the first leg, or $6.0 \cdot 10^{15}$ meters, and the total time elapsed on earth during the rocket's journey is four times that elapsed during the first leg, or $4 \cdot 84 = 336$ years. So it should be the year 2436 on earth by the time the rocket returns home.

Problem 11.13

An infinitely long straight wire of negligible cross-sectional area is at rest and has a uniform linear charge density q_0 in the inertial frame K'. The frame K' (and the wire) move with a velocity $\mathbf{v}$ parallel to the direction of the wire with respect to the laboratory frame.

(a) Write down the electric and magnetic fields in cylindrical coordinates in the rest frame of the wire. Using the Lorentz transformation properties of the fields, find the components of the electric and magnetic fields in the laboratory.

(b) What are the charge and current densities associated with the wire in its rest frame? In the laboratory?

(c) From the laboratory charge and current densities, calculate directly the electric and magnetic fields in the laboratory. Compare with the results of part (a).

I don't like q_0 as a symbol for charge density, because it appears to have the wrong units. I'll use λ instead. We'll take our z axis to coincide with the wire and take $\mathbf{v}$ in the positive z direction.

(a) In the rest frame there is no current and the $\mathbf{E}$ field is static; hence $\mathbf{B} = 0$. The electric field is found by considering a Gaussian pillbox in the shape of a right circular cylinder coaxial with the wire and of radius r and length dz. There is no electric field normal to the upper and lower surfaces, and the field normal to the radial bounding surface is uniform across the circumference. On the other hand, the charge enclosed in the cylinder is $\lambda\, dz$. Then Gauss' law is

$$\begin{aligned} \oint \mathbf{E} \cdot d\mathbf{A} &= 4\pi Q \\ \Longrightarrow 2\pi r dz E_r &= 4\pi\lambda dz \\ \Longrightarrow E_r &= \frac{2\lambda}{r}. \end{aligned} \tag{9}$$

In terms of cartesian components we have

$$E_x = 2\lambda\left(\frac{x'}{x'^2 + y'^2}\right), \qquad E_y = 2\lambda\left(\frac{y'}{x'^2 + y'^2}\right).$$

The field-strength tensor in the laboratory frame is

$$
\begin{aligned}
F &= \Lambda F' \tilde{\Lambda} \\
&= 2\lambda \frac{1}{x'^2 + y'^2} \begin{pmatrix} \gamma & 0 & 0 & \beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \beta\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} 0 & -x' & -y' & 0 \\ x' & 0 & 0 & 0 \\ y' & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \gamma & 0 & 0 & \beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \beta\gamma & 0 & 0 & \gamma \end{pmatrix} \\
&= \frac{2\gamma\lambda}{(x'^2 + y'^2)} \begin{pmatrix} 0 & -x' & -y' & 0 \\ x' & 0 & 0 & \beta x' \\ y' & 0 & 0 & \beta y' \\ 0 & -\beta x' & -\beta y' & 0 \end{pmatrix}.
\end{aligned}
$$

Reading off from this the transformed fields, and dropping the primes on x and y since the z boost leaves these coordinates unchanged, we have

$$
\begin{aligned}
\mathbf{E} &= \gamma \frac{2\lambda}{x^2 + y^2} (x\mathbf{i} + \mathbf{y}\mathbf{j}) && (10) \\
&= \gamma \frac{2\lambda}{r} \hat{\mathbf{r}} && (11) \\
\mathbf{B} &= \beta\gamma \frac{\lambda}{2(x^2 + y^2)} (-y\mathbf{i} + x\mathbf{j}) && (12) \\
&= \beta\gamma \frac{2\lambda}{r} \hat{\phi}. && (13)
\end{aligned}
$$

(b) In the rest frame there is no current and the charge density is $\rho' = \lambda\delta(x)\delta(y)$, so

$$\mathcal{J}'^{\mu} = c\lambda(\delta(x)\delta(y), 0, 0, 0).$$

The transformed current density is

$$
\begin{aligned}
\mathcal{J}^{\mu} &= \Lambda^{\mu}_{\nu} \mathcal{J}'^{\nu} \\
&= c\lambda \begin{pmatrix} \gamma & 0 & 0 & \gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \beta\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} \delta(x)\delta(y) \\ 0 \\ 0 \\ 0 \end{pmatrix} \\
&= c\gamma\lambda \begin{pmatrix} \delta(x)\delta(y) \\ 0 \\ 0 \\ \beta\delta(x)\delta(y) \end{pmatrix}.
\end{aligned}
$$

(c) Computing the electric field in the laboratory frame is easy, since the charge density is the same as we had before but with a factor of γ thrown in. Then the $\mathbf{E}$ field is just (9) but with that factor of γ thrown in, i.e.

$$\mathbf{E} = \frac{2\gamma\lambda}{r} \hat{\mathbf{r}}$$

which agrees with (10).

For the magnetic field, we note that the current density in the lab frame is $\mathbf{J} = c\beta\gamma\lambda\delta(x)\delta(y)\hat{\mathbf{z}}$. Then the current piercing a disc of radius r is $I = c\beta\gamma\lambda$. On the other hand, by symmetry the magnetic field in the azimuthal direction around the circumference of this disc is constant, so we may use Ampere's law, $\nabla \times \mathbf{B} = (4\pi/c)\mathbf{J}$ to write

$$2\pi r B_\phi = \frac{4\pi}{c}(c\beta\gamma\lambda)$$

or

$$\mathbf{B} = \frac{2\beta\gamma\lambda}{r}\hat{\phi}$$

which agrees with (13).

Problem 11.15

> In a certain reference frame a static, uniform, electric field E_0 is parallel to the x axis, and a static, uniform, magnetic induction $B_0 = 2E_0$ lies in the $x-y$ plane, making an angle θ with the axis. Determine the relative velocity of a reference frame in which the electric and magnetic fields are parallel. What are the fields in that frame for $\theta \ll 1$ and $\theta \to (\pi/2)$?

The untransformed fields are

$$\mathbf{E} = E_0\mathbf{i}, \qquad \mathbf{B} = 2E_0(\cos\theta\mathbf{i} + \sin\theta\mathbf{j}).$$

Let's suppose we boost along the z axis. Then the fields transform according to Jackson equation (11.149):

$$\begin{aligned}
\mathbf{E}' &= \gamma E_0(1 - 2\beta\sin\theta)\mathbf{i} + 2\gamma\beta E_0\cos\theta\mathbf{j} \\
\mathbf{B}' &= 2\gamma E_0\cos\theta\mathbf{i} + \gamma E_0(2\sin\theta + \beta)\mathbf{j}.
\end{aligned}$$

The angle between the fields is given by

$$\begin{aligned}
\cos\theta &= \frac{\mathbf{E}'\cdot\mathbf{B}'}{|\mathbf{E}'||\mathbf{B}'|} \\
&= \frac{2\gamma^2 E_0^2\cos\theta(1+\beta^2)}{\gamma^2 E_0^2(1 - 4\beta\sin\theta + 4\beta^2)^{1/2}(4 + 4\beta\sin\theta + \beta^2)^{1/2}}
\end{aligned}$$

Setting this equal to unity, we obtain

Problem 11.18

The electric and magnetic fields of a particle of charge q moving in a straight line with speed $v = \beta c$, given by (11.152), become more and more concentrated as $\beta \to 1$, as indicated in Fig. 11.9. Choose axes so that the charge moves along the z axis in the positive direction, passing the origin at $t = 0$. Let the spatial coordinates of the observation point be (x, y, z) and define the transverse vector $\mathbf{r}_\perp$, with components x and y. Consider the fields and the source in the limit of $\beta = 1$.

(a) Show that the fields can be written as

$$\mathbf{E} = 2q\frac{\mathbf{r}_\perp}{r_\perp^2}\delta(ct - z); \qquad \mathbf{B} = 2q\frac{\hat{\mathbf{v}} \times \mathbf{r}_\perp}{r_\perp^2}\delta(ct - z).$$

(b) Show by substitution into the Maxwell equations that these fields are consistent with a 4-vector source density,

$$J^\alpha = qcv^\alpha \delta^{(2)}(\mathbf{r}_\perp)\delta(ct - z)$$

where the 4-vector $v^\alpha = (1, \hat{\mathbf{v}})$.

(c) Show that the fields of part a are derivable from either of the following 4-vector potentials,

$$A^0 = A^z = -2q\delta(ct - z)\ln(\lambda r_\perp); \qquad \mathbf{A}_\perp = 0$$

or

$$A^0 = A^z = 0; \qquad \mathbf{A}_\perp = -2q\Theta(ct - z)\nabla_\perp \ln(\lambda r_\perp)$$

where λ is an irrelevant parameter setting the scale of the logarithm. Show that the two potentials differ by a gauge transformation and find the gauge function, χ.

(a) In the reference frame in which the particle is at rest at the origin, the fields are

$$\mathbf{E}' = \frac{q}{(x'^2 + y'^2 + z'^2)^{3/2}}(x'\mathbf{i} + y'\mathbf{j} + z'\mathbf{k}), \qquad \mathbf{B}' = 0.$$

Transforming back to the laboratory frame according to Jackson 11.148, the electric field is

$$\mathbf{E} = \frac{q}{(x'^2 + y'^2 + z'^2)^{3/2}}(\gamma x'\mathbf{i} + \gamma y'\mathbf{j} + z'\mathbf{k})$$

where in this expression the coordinates are still those of the observation point in the moving frame. The transformation of these to the lab frame is $x' = x$, $y = y' = y$, $z' = \gamma(z - \beta ct)$. Then the correct expression for the transformed field is

$$\mathbf{E} = \frac{q}{[x^2 + y^2 + \gamma^2(z - ct)^2]^{3/2}}(\gamma x\mathbf{i} + \gamma y\mathbf{j} + (z - ct)\mathbf{k}).$$

In the limit $\beta \to 1$, we have $\gamma \to \infty$. For $z \neq ct$ the γ^2 factor in the denominator then ensures that all field components are zero. For $z = ct$, however, although the z component of the field clearly vanishes, the behavior of the other components is not as immediately clear. To elucidate the behavior of, say, the x component of the field at $z = ct$ we integrate it from $z = ct - \epsilon$ to $z = ct + \epsilon$:

$$\begin{aligned}
\int_{ct-\epsilon}^{ct+\epsilon} E_x dz &= q\gamma x \int_{ct-\epsilon}^{ct+\epsilon} \frac{dz}{[r_\perp^2 + \gamma^2 (z-ct)^2]^{3/2}} \\
&= \frac{qx}{r_\perp^2} \int_{-\gamma\epsilon/r_\perp}^{\gamma\epsilon/r_\perp} \frac{du}{[1+u^2]^{3/2}} \\
&= \frac{2qx}{r_\perp^2} \frac{\gamma\epsilon}{\sqrt{r_\perp^2 + \gamma^2\epsilon^2}}.
\end{aligned}$$

Taking the limit $\gamma \to \infty$ for any finite ϵ, we find

$$\lim_{\gamma\to\infty} \int_{ct-\epsilon}^{ct+\epsilon} E_x dz = \frac{2qx}{r_\perp}^2. \tag{14}$$

On the other hand, integrating between two points on the same side of $z = ct$, say from $z = ct + \epsilon$ to $z = ct + 2\epsilon$, we find

$$\int_{ct+\epsilon}^{ct+2\epsilon} E_x dz = \frac{2qx}{r_\perp}^2 \left[\frac{2\gamma\epsilon}{\sqrt{r_\perp^2 + 4\gamma^2\epsilon^2}} - \frac{\gamma\epsilon}{\sqrt{r_\perp^2 + \gamma^2\epsilon^2}} \right]$$

which vanishes as $\gamma \to \infty$.

Since E_x vanishes at any point $z \neq ct$ but yields something nonzero when integrated across that point, we conclude that it is just a δ function in $(z - ct)$ with coefficient given by (14):

$$E_x = \frac{2qx}{r_\perp^2} \delta(z - ct).$$

and, similarly,

$$E_y = \frac{2qy}{r_\perp^2} \delta(z - ct).$$

Combining these, we can write

$$\mathbf{E} = 2q \frac{\mathbf{r}_\perp}{r_\perp^2} \delta(ct - z).$$

The $\mathbf{B}$ field is given by Jackson (11.150) with, in the ultrarelativistic limit, $\beta = \hat{\mathbf{k}}$:

$$\mathbf{B} = 2q \frac{\hat{\mathbf{v}} \times \mathbf{r}_\perp}{r_\perp^2} \delta(ct - z).$$